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Several risk measures in portfolio selection: Is it worthwhile? *

¿Está justificado el uso de varias medidas de riesgo en la selección de carteras?

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ABSTRACT This paper is concerned with asset allocation using a set of three widely used risk measures, which are the variance or deviation, Value at Risk and the Conditional Value at Risk. Our purpose is to evaluate whether solving the asset allocation problem under several risk measures is worthwhile or not, given the added computational complexity. The main contribution of the paper is the solution of two models that consider several risk measures: the *mean-variance-VaR* model and the *mean-VaR-CVaR* model. The inclusion of *VaR* as one of the objectives to minimize leads to nonconvex problems, therefore the approach we propose is based on a heuristic: multi-objective genetic algorithms. Our results show the adequacy of the multi-objective approach for the portfolio optimization problem and emphasize the importance of dealing with *mean-σ-VaR* or *mean-VaR-CVaR* models as opposed to *mean-σ-CVaR*, where both risk measures are redundant.

KEYWORDS Portfolio Selection; Value-at-Risk; Conditional Value-at-Risk; Multi-Objective Decision Making.

RESUMEN Este articulo aborda el problema de selección de carteras empleando tres medidas de riesgo ampliamente utilizadas: varianza o desviación típica, Valor en Riesgo *(VaR)* y Valor en Riesgo Condicional *(CVaR)*. Nuestro principal objetivo es evaluar la relevancia de incluir simultáneamente varias medidas del riesgo, dada la complejidad computacional que supone. La principal contribución de este artículo es la propuesta de solución de dos modelos que consideran simultáneamente dos medidas del riesgo muy utilizadas: el modelo de *media-varianza-VaR* y el modelo *media-VaR-CVaR*. La inclusión del *VaR* como uno de los objetivos a minimizar convierte el problema de optimización en no convexo, por lo que el método de resolución propuesto está basado en una heurística muy actual: algoritmo genético multiobjetivo. Nuestros resultados muestran la adecuación del enfoque multiobjetivo para resolver el problema de optimización de carteras y justifica y enfatiza la importancia de emplear los modelos *media-varianza-VaR* o *media-VaR-CVaR* en lugar del modelo *media-varianza-CVaR*, en el que las dos medidas de riesgo simultáneas resultan redundantes.

PALABRAS CLAVE Selección de carteras; Valor en riesgo; Valor en riesgo condicional; Toma de decisiones multiobjetivo.

1. INTRODUCTION

Portfolio selection is obtained maximizing expected return and minimising risk. There are several ways of measure risk of a portfolio. The classical measure of risk is the variance or

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standard deviation used by seminal work of Markowitz (1952). In this case, the portfolio selection is done by solving a quadratic problem. This risk measure weights equally positive against negative returns. This is justified by assuming either that investors have quadratic utility functions or that asset returns are drawn from a multivariate elliptical distribution. Other measures known as downside risk measures have been proposed to capture the left-hand side of a return distribution which involves risks. Bawa (1975) and Fisburn (1977) introduce a general definition of downside risk in the form of lower partial moments. An example is semi-variance, which is a special case of lower partial moments.

Nowadays researchers and practitioners are focused on *Value-at-Risk (VaR)* and Conditional *Value-at-Risk (CVaR)* as measures of market risk. *VaR* of a portfolio is the lowest amount such that the loss will not exceed it with probability 1-α. *CVaR* is the conditional expectation of losses above the *VaR*. *VaR* and *CVaR* can be used to balance risk and return. While *CVaR* can be efficiently minimized using linear programming and non-smoothing techniques (Rockafellar and Uryasev, 2000), minimizing *VaR* leads to a non-convex and non-differential risk-return problem and smoothing techniques (Gaivoronski and Pflug, 2005) or heuristic optimization techniques need to be applied (Gilli *et al.,* 2006). Although variance is still the most widely used measure of risk in the practice of portfolio selection, *VaR* and *CVaR* are used as risk limit and to control risk by the fund management industry. This has motivated the inclusion of *VaR* and *CVaR* as constraints in the classical mean-variance problem (Alexander *et al.,* 2007).

Each risk measure, variance, *VaR* and *CVaR*, captures different aspects of risk. Therefore, it could be worthy to introduce them jointly in the portfolio analysis. In the literature we can find some works that explore this idea. For instance, Roman *et al.,* (2007) proposed the use of two risk measures (variance and *CVaR*) in order to perform a portfolio selection. The mean-variance-*CVaR* model proposed is a multi-objective problem. A method to obtain all the efficient solutions is given and tested on the solutions of mean-variance model and mean-*CVaR* model.

This paper contributes to the literature in several ways. Firstly, it empirically shows the validity of multiobjective genetic algorithms for obtaining efficient portfolios under different risk measures including VaR. Secondly, even though normality is not assumed, we show that variance and *CVaR* identify similar efficient portfolios and thirdly, we show how multiobjective genetic algorithms can deal with three explicit objective functions: maximize return and minimize two risk measures without making assumptions about the asset return distribution. The proposal is validated on real data from Eurostoxx 50 index.

The structure of the paper is as follow: Section 2 formulates the problems under one and several risk measures. Section 3 presents the multiobjective genetic algorithm proposed. Data and empirical results are presented in Section 4. Finally, in Section 5 some conclusions are drawn.

2. THE PORTFOLIO SELECTION PROBLEM UNDER A SET OF RISK MEASURES

Although investors may propose different risk measures, the three stages of Markowitz framework are still valid: the stage of security analysis, which comprises the collection of data and the estimation of relevant parameters; the stage of portfolio analysis, where the efficient set of portfolios is computed by solving an optimization problem dependant on the risk measure; and the stage of portfolio choice, which entails the selection of an optimal portfolio depending on the specific preferences of the investor.

2.1. Risk measures

The choice of a measure of risk plays a fundamental role for the portfolio optimization problem. Following Artzner *et al.* (1999), a risk measure can be viewed as a single number $\rho(r)$ assigned to the distribution of the portfolio return r. This risk measure should satisfy four properties: monotonicity, translation invariance, homogeneity and subadditivity.

Monotonocity: if $r_1 \leq r_2$ then $\rho(r_1) \geq \rho(r_2)$, that is, if a portfolio has systematically lower returns than another for all states of the world, its risk must be greater.

Traslation invariance: $\rho(r+k) = \rho(r) - k$, that is, adding cash k to a portfolio should reduce its risk by k.

Homogenity: $\rho(br) = b\rho(r)$, that is, increasing the size of a portfolio by b should scale its risk by the same factor.

Subadditivity: $\rho(r_1 + r_2) \leq \rho(r_1) + \rho(r_2)$, that is, merging portfolios cannot increase risk.

Unfortunately, there are several risk measures that satisfy these properties. We can identify, at least, three widely used risk measures: variance, *VaR* and *CVaR*. Variance and *CVaR* satisfy the four properties and *VaR* satisfies three of them. *VaR* does not satisfies subadditivity under certain conditions.

2.1.1. *Variance*

Variance or standard deviation of returns is a traditional risk measure which measure volatility. In Markowitz formulation risk is defined as the variance of returns, that is the weighted sum of squared deviations around the mean. Therefore risk is measured as the dispersion of possible outcomes. A flatter distribution indicates grater risk, and a tighter distribution, lower risk.

If R_j $j = 1,..T$ are historical observations of r, the expected return is estimated by the sample mean, $\bar{R} = \frac{1}{T}\sum_{j=1}^{T} R_j$, the variance is estimated by the sample variance, $\sigma^2 = \frac{1}{T}\sum_{j=1}^{T}(R_j - \bar{R})^2$, and the squared root of σ^2 is the standard deviation of rates of return. Variance treats symmetrically positive or negative deviations from the mean and, when returns are asymmetric, it can give a partial picture of risk.

2.1.2. *VaR*

The quantity identifies, with a certain confidence interval, the maximum anticipated loss in portfolio due to adverse market movements. It is a compact representation of risk level and measures downside risk. By definition *VaR* is a quantile of the probability distribution of the portfolio value. Let *f(r)* the probability distribution function of the future portfolio return and α the significant level (usually 5%), *VaR* is implicitly defined in the following equation, $\int_{-\infty}^{Var} f(r)dr = \alpha$, and using the cumulative distribution function *F(r)* can be explicitly defined as, $VaR_\alpha = inf\left\{ r \,|\, F(r) \geq \alpha \right\}$.

The *VaR* computation can be simplified considerably if the distribution *f(r)* can be assumed to belong to a parametric family, such as the normal distribution. In this case, the *VaR* can be derived directly from the portfolio standard deviation using a multiplicative factor that depends on the confidence level. For example, to find *VaR* for 5% left-tail significant level of a normal variable corresponds to a -1.645 times the standard deviation since -1.645 is the quantile of the standard normal distribution. Hence, *VaR* is simply a multiple of the standard deviation which is related to the confidence level. Under this assumption, *VaR* and variance are interchangeable so independently of using *VaR* or variance we obtain the same efficient portfolio. The issue is which distribution may fit the data. It is not trivial, an approach to test the conditional distribution can be found in Baixauli and Alvarez (2004).

In this work we compute *VaR* using the historical simulation method, which is perhaps the most widely used method to compute *VaR*. It consists of going back in time and applying current weights to a time-series of historical asset returns. By keeping weights at their current values the history of a hypothetical portfolio is reconstructed.

$$R_{p,j} = \sum_{i=1}^{N} w_i R_{i,j} \qquad \text{for } j = 1,\ldots,T \tag{1}$$

VaR is obtained from the entire distribution of hypothetical returns, where each historical day is assigned the same probability of occurrence *(1/T)*. By sorting in ascending order $R_{p,j}$ we choose the $R^*_{p,j}$ in the position αT.

This method is simple to implement, it short-circuits the need for an estimation of a covariance matrix, it simplifies the computations and by relying on actual prices, it allows non-linearities and non-normal distributions. This method captures correlations and it does not rely on specific assumptions about valuation models or the underlying stochastic structure of the market. Perhaps most important, it can account for fat tails and asymmetries. Under the historical simulation model *VaR* and variance are not interchangeable and the efficient frontier could, and probably will, be different.

In Artzner *et al.* (1999) the *VaR* measure was criticized due to the fact that it does not always satisfy subadditivity, one of the four properties for a risk measure to be coherent. However, Basel Committee (2001) assumes VaR as a risk measure and the regulatory capital for a loan is correlated to its marginal contribution to VaR. Finally, *VaR* is a non-convex risk measure, which makes it difficult to use in optimization problems.

2.1.3. *CVaR*

CVaR, for continuous distributions also known as the mean excess loss, mean shortfall and tail *VaR*, is defined as the average of all losses exceeding the *VaR* and it is computed as the expected value of *r* conditional on exceeding the *VaR*,

$$CVaR_{\alpha} = \frac{\int_{-\infty}^{Var} rf(r)dr}{\int_{-\infty}^{Var} f(r)dr} \tag{2}$$

This risk measure informs about how much we could lose if the portfolio return falls beyond *VaR*. Rockafellar and Uryasev (2000) proposed *CVaR* as a risk measure which overcomes the problems posed by the use of *VaR* for capital allocation. *CVaR* satisfies the four properties and it is considered a coherent risk measure. However, it is not as popular as *VaR* since financial institutions have promoted *VaR* as standard measure of market risk.

Also, *CVaR* computation can be simplified if the distribution *f(r)* is assume to be normal. In this case, *CVaR* can be expressed as follows:

$$CVaR_{\alpha} = \frac{-\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(VaR - \mu)^2}{2\sigma^2}}}{\alpha} \tag{3}$$

CVaR captures the shape of the tail of the distribution *f(r)* and when we assume normality we are restricting this shape to a functional form. For example, in a standard normal distribution, given the significant level of 5%, *VaR* is -1.645 and *CVaR* is -2.062. Again, under this assumption *VaR*, *CVaR* and variance are interchangeable in a portfolio optimization problem since the value of these risk measure for different portfolios gives the same ranking, that is, if a portfolio *A* has a variance greater than portfolio *B* then *VaR* of portfolio *A* will be greater than *VaR* of portfolio *B* and *CVaR* of portfolio *A* will be greater than *CVaR* of portfolio *B*. Therefore when a return distribution is assumed to be normal the mean-variance model, *mean-CVaR* model and *mean-VaR* model to solve for obtaining the efficient set of portfolios are equivalent.

In this work we compute *CVaR* using the historical simulation method as in the case of *VaR*, that is, we obtain *CVaR* from the entire distribution of historical returns, as the sample mean of $R_{p,j}$ lower than *VaR*

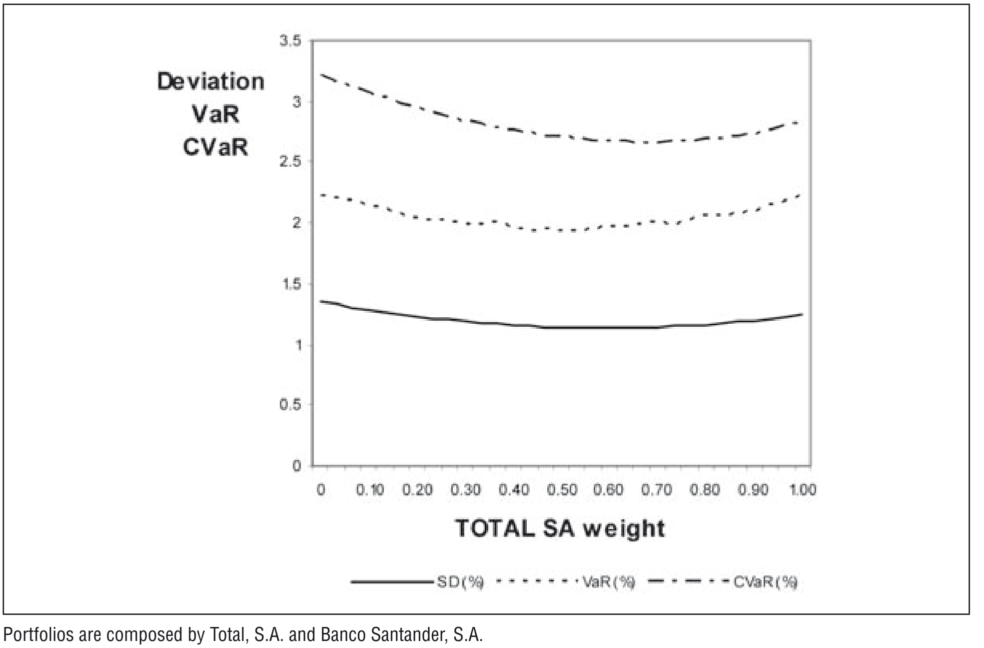
$$CVaR_{\alpha} = \frac{\sum_{j=1}^{\alpha T} R_{p,j}}{\alpha T} \tag{4}$$

Obtaining *CVaR* by historical simulation method has the same advantages than using historical simulation for calculating *VaR*. It is simple to implement and it allows non-normal distribution. Under the historical simulation model *CVaR*, *VaR* and variance are not interchangeable and the efficient frontier could be different.

Variance and *CVaR* are convex risk measures. The convexity of an objective function is an essential point for demonstrating the convergence of numerical optimization methods because for convex functions every local minimum is a global one and the set of optimal strategies is convex. However, *VaR* is a non-convex risk measure and it makes it difficult to use in optimization problems.

In order to show the irregularity of *VaR* against standard deviation and *CVaR,* Figure 1 from Baixauli *et al.* (2010) shows the *VaR*, standard deviation and *CVaR* for all feasible portfolios composed of two stocks (Total, S.A. and Banco Santander) computed with historical simulation method. The *x-axis* shows the percentage of the budget invested in Total, S.A. while one minus this value is the percentage invested in Banco Santander. Figure 1 shows the convexity of the standard deviation and *CVaR* with a global minimum whereas *VaR* is not a convex function and it presents several local minimums. Moreover, the standard deviation, *VaR* and *CVaR* are not interchangeable risk measures because the minimum deviation portfolio and the minimum *VaR* portfolio do not coincide. Total, S.A. weight in the optimal portfolio using deviation is 59%, using *VaR* is 49% and using *CVaR* is 68%.

FIGURE 1
DEVIATION, *VAR* AND *CVAR* FOR PORTFOLIOS WITH TWO ASSETS



Portfolios are composed by Total, S.A. and Banco Santander, S.A.

Obviously, this is only an example of a portfolio composed by two securities but it illustrates the issue that we address in this paper with n securities.

2.2. Portfolio selection under one risk measure

When a new risk measure, such *VaR* and *CVaR* is proposed the computational complexity of the optimization problem can increase. Hence, the problem to solve in the stage of portfolio analysis varies. While variance can be minimized using a quadratic problem and *CVaR* can be minimized using a linear problem following Rockafellar and Uryasev (2000), the use of *VaR* leads to a *non-convex NP-hard* optimization problem.

In order to evaluate convergence of multiobjective genetic algorithm, we formulate and compute mean-variance, *mean-VaR* and *mean-CVaR* as multiobjective problems, and also mean-variance model as a quadratic problem and *mean-CVaR* model as a linear problem.

An important property of the variance of the portfolio return is that it allows defining the objective function as a quadratic function. The mean-variance model can be formulated as follows:

$$
\begin{array}{ll}
\textit{Model 1A} & \textit{Model 1B} \\
\underset{w}{Min}\ \sigma^2(w) = w'\Omega w & \underset{w}{Min}\ \sigma^2(w) = w'\Omega w \\
s.t. & \underset{w}{Max} E(r) = w'r \\
w'r \geq r^* & s.t. \\
w'\mathbf{1} = 1 & w'\mathbf{1} = 1 \\
w_j \geq 0\ \forall_j = 1,\ldots,n & w_j \geq 0\ \forall_j = 1,\ldots,n
\end{array}
\tag{5}
$$

where w is a vector containing the percentage of the budget invested in each asset, Ω is the covariance matrix and r is a vector containing the expected rate of return of each asset. Model 1*A* is the classical quadratic problem whose exact solution is a function of the expected return fixed by the investor, r^*, while Model 1*B* is the multiobjective problem that we solve with a multiobjective genetic algorithm.

The *mean-VaR* model uses the concept of *VaR* as a measure of risk instead of variance. Then, efficient portfolios are the solution of the following problem:

Model 2

$$
\begin{array}{l}
\underset{w}{Min}\ VaR_\alpha(w) = inf\left\{r \mid F(r) \geq \alpha\right\} \\
\underset{w}{Max}\ E(r) = w'r \\
s.\ t. \\
w'\mathbf{1} = 1 \\
w_j \geq 0\ \forall j = 1,\ldots,n
\end{array}
\tag{6}
$$

VaR is difficult to optimize for discrete distributions since is non-convex and has multiple local extrema. Mostly, approaches rely on linear approximation of the portfolio risks and assume a joint normal distribution of the underlying parameters (Jorion, 1996, Duffie and Pan, 1997). Optimization requires smoothing or heuristic techniques as those presented in Gaivoronski and Pflug (2005) or Gilli *et al.* (2006). We use a multiobjective genetic algorithm which does not rely on specific assumptions about the distribution of the portfolio return.

As we pointed out previously, *CVaR* can be defined as the conditional expected loss under the condition that it exceeds *VaR*. For general distributions, *CVaR* is more attractive than VaR since it is sub-additive and convex. Hence, the problem of minimizing *CVaR* for finding efficient portfolios is convex. *CVaR* of a random variable *r(w,y)*, which represents the return of the portfolio *w* under scenarios *y*, can be calculated by solving a convex optimization problem (Rockafellar and Uryasev, 2000, 2002). The approach characterises $VaR_\alpha(w)$ and $CVaR_\alpha(w)$ in terms of the function $F_\alpha(w, VaR)$,

$$F_\alpha(w,VaR) = \text{-}VaR = \frac{1}{\alpha}\int\left[-r(w,y)+VaR\right]^{+}p(y)dy \tag{7}$$

Where $[u]^+ = u$ for $u \geq 0$ and $[u]^+ = 0$ for $u \leq 0$. As a function of $VaR_\alpha(w)$, $F_\alpha(w, VaR)$ is convex and continuously differentiable and $CVaR_\alpha(w)$ can be determined minimizing $F_\alpha(w, VaR)$, that is, $CVaR_\alpha(w) = \min F_\alpha(w, VaR)$. The integral in (7) can be approximated by sampling the probability distribution of y according to its density *p(y)*. If sampling generates *T* scenarios, $y_j\, j = 1,\ldots,T$ then the approximation is,

$$F_\alpha(w,VaR) = -VaR + \frac{1}{\alpha}\sum_{j=1}^{T}\pi_j\left[-r(w,y_j)+VaR\right]^{+} \tag{8}$$

where π_j is the probability of scenarios y_j and $r(w,y_j) = \sum_{i=1}^{n} w_j r_{ij}$, being r_{ij} the return of asset *i* under scenario *j*. Using auxiliaries variables z_j, $j = 1,\ldots,T$ the function $F_\alpha(w, VaR)$ can be replaced by the linear function, $F\alpha(w,VaR) = -VaR + \frac{1}{\alpha}\sum_{j=1}^{T}\pi_j z_j$ and the set of linear constraints, $z_j \geq -r(w,y_j) + VaR$, $z_j \geq 0, j = 1,\ldots,T$. Then investors have to solve the linear problem represented in model 3*A*. Lim *et al.* (2008) propose an approach to reduce the time consumed for solving the linear problem under a large number of scenarios.

Model 3*A*

$$\begin{aligned}
&MinCVaR_\alpha(w) = \underset{VaR,w,z}{Min}\ \text{-}VaR + \frac{1}{\alpha T}\sum_{j=1}^{T} z_j \\
&s.\ t. \\
&z_j \geq -\sum_{j=1}^{T} w_j r_{ij} + VaR \quad \forall j = 1,\ldots,T \\
&z_j \geq 0\ \forall j = 1,\ldots,T \\
&w'r \geq r^* \\
&w'\mathbf{1} = 1 \\
&w_j \geq 0\ \forall j = 1,\ldots,n
\end{aligned}$$

Model 3*B*

$$\begin{aligned}
&MinCVaR_\alpha(w) = \frac{1}{\alpha T}\sum_{j=1}^{\alpha} R_{p,j} \\
&\underset{w}{Max}\ E(r) = w'r \\
&s.\ t. \\
&w'\mathbf{1} = 1 \\
&w_j \geq 0\ \forall j = 1,\ldots,n
\end{aligned} \tag{9}$$

Model 3*A* is a linear problem with an exact solution for each r^* and Model 3*B* is the formulation used in the multiobjective algorithm problem. In this case we do not need linearization since a multiobjective genetic algorithm searches the space of solutions.

Solving Model 1 (*A* or *B*), 2 and 3 (*A* or *B*) allows us to obtain sets of optimal portfolios. The question is how different the efficient portfolios obtained with different risk measures are. Are portfolios that are efficient in the mean variance model (Model 1*A* or 1*B*) efficient in *mean-VaR* or *mean-CVaR* space or not? If the answer is yes or next to yes, given that increasing the number of risk measures implies defining the investor's preferences through more complex utility functions, then we should use only one risk measure. However, if the answer is not, that is, if each risk measure reflects different risk dimensions of a portfolio then we should consider the differences between the efficient frontiers increasing the number of risk measures. In this case it is necessary to formulate the selection problem as a multiobjective problem with two or more risk measures to minimize.

2.3. Portfolio selection under several risk measures

The inclusion of more than one risk measure to obtain efficient portfolios has been approached in the literature from two perspectives: introducing additional constrains or introducing additional objective functions as multiobjective problems. Early examples on using an additional variable for the mean-variance model can be found in Konno *et al.* (1993) in which mean, absolute deviation and skewness are combined. The skewness is maximized subject to constraints in mean and absolute deviation. Konno and Suzuki (1995) show an efficient algorithm to optimize a mean-variance-skewness model. Alexander *et al.* (2007) obtain efficient portfolios under mean-variance model when *VaR* or *CVaR* are used as constraints. A pure multiobjective proposal can be found in Roman *et al.* (2007) where a *mean-variance-CVaR* model is proposed and an optimization approach is given. Introducing several risk measures provides more information about portfolio risk. Although the asset return was non-normal distributed, portfolio return follows other unknown distribution and efficient portfolios are a subset of all feasible portfolios and their unknown distribution can behave similar to the normal distribution. Under normal distributions of returns the three measures provide the same optimal portfolio (Rockafellar and Uryasev, 2000) and a multiobjective risk is worthless. However, for skewed distributions, *VaR* and *CVaR* optimal portfolios may be quite different.

A multiobjective problem in general is expressed as $Max\{f_1(w), f_2(w), \ldots, f_n(w)\}$. In the classical asset allocation problem (model 1*B*) there are two objectives to optimize: mean and variance. If an additional risk measure is introduced the concept of efficient portfolio is the same but the problem has a third dimension. A feasible solution w^* Pareto dominates another feasible solution w' if $f_i(w^*) \geq f_i(w')$ for all i with at least one strict inequality. For instance, in a *mean-VaR-CVaR* model the efficient portfolios are the Pareto efficient solutions of a multiobjective problem in which the expected value is maximized while the *VaR* and *CVaR* are minimized. The problem may be formulated as, Model 4

$$
\begin{aligned}
&\underset{w}{Max}\ E(r) = w'r \\
&\underset{w}{Min} VaR_{\alpha}(w) = inf\{r \mid F(r) \geq \alpha\} \\
&\underset{w}{Min} CVaR_{\alpha}(w) = \frac{1}{\alpha T}\sum_{j=1}^{\alpha} R_{p,j} \\
&s.\ t. \\
&w'\mathbf{1} = 1 \\
&w_j \geq 0\ \forall j = 1,\dots, n
\end{aligned}
\tag{10}
$$

Minimizing *VaR* does not control losses exceeding *VaR* since it does not take into account the shape of the tail. Then, introducing *CVaR* as a second measure of risk captures these losses. In *mean-VaR-CVaR* model (Model 4), w^* is preferred to portfolio w' if and only if $E(r_{w^*}) \geq E(r_{w'})$, $VaR_{\alpha}(w^*) \leq VaR_{\alpha}(w')$, and $CVaR_{\alpha}(w^*) \leq CVaR_{\alpha}(w')$ with at least one strict inequality. When we consider three objectives we obtain a surface of efficient portfolios instead of a line. One issue addressed in this paper is how the efficient frontier changes when we increase the measures of risk.

On the other hand, in a *mean-variance-VaR* model the efficient portfolios are the Pareto efficient solutions of:

Model 5

$$
\begin{aligned}
&\underset{w}{Max}\ E(r) = w'r \\
&\underset{w}{Min}\ \sigma^2(w) = w'\Omega w \\
&\underset{w}{Min}\ VaR_{\alpha}(w) = inf\{r \mid F(r) \geq \alpha\} \\
&s.\ t. \\
&w'\mathbf{1} = 1 \\
&w_j \geq 0\ \forall j = 1,\dots, n
\end{aligned}
\tag{11}
$$

This paper is related to Roman *et al.* (2007), which study efficient portfolios considering more than one measure of risk as objective function. The main difference of our contribution is that we present a practical optimization approach where *VaR* is considered as a measure of risk in a multiobjective problem. In addition, we evaluate if the inclusion of several risk measures is needed or irrelevant for efficient portfolios. As Roman *et al.* (2007) indicates *VaR* leads to complex problems, whose solution requires the development of new algorithms. Roman *et al.* (2007) only use *mean-variance-CVaR* model while we focus on *mean-VaR-CVaR* (Model 4) and *mean-variance-VaR* (Model 5) models.

3. OPTIMIZATION APPROACH: MULTIOBJECTIVE GENETIC ALGORITHM

By definition, the portfolio optimization problem proposed by Markowitz (1952) is multi-objective with two conflicting criteria: maximising the return and minimising the level of risk. Introducing several risk measures increases the number of objectives. Multi-objective optimization problems do not have a single solution but a set of solutions equally optimal that define the efficient frontier (or Pareto-optimal front).

Genetic algorithms *(GAs)* (see, Holland, 1975, Goldberg, 1989) are stochastic optimization techniques that mimic the way species evolve in nature. *GAs* emulate this process by encoding the points of the search space (called individuals) in a chromosome-like shape and evolving a population of them through a number of generations using mechanisms drawn from natural evolution. Along the evolution the individuals' fitness is evaluated according to how well they solve the problem at hand. The better the fitness of an individual, the more chances it has to produce offspring for the next generation. As the generations progress, it results in the prevalence of stronger solution over weaker ones. Thus, the evolution process tends to near optimal solutions.

A *GA* initiates the process of searching by randomly generating an initial population of possible solutions. The performance of each solution is evaluated using a fitness function, which is a measure of how good the performance of the solution is. Then, a new generation is produced according to the three main operators of the *GA*: selection, crossover and mutation.

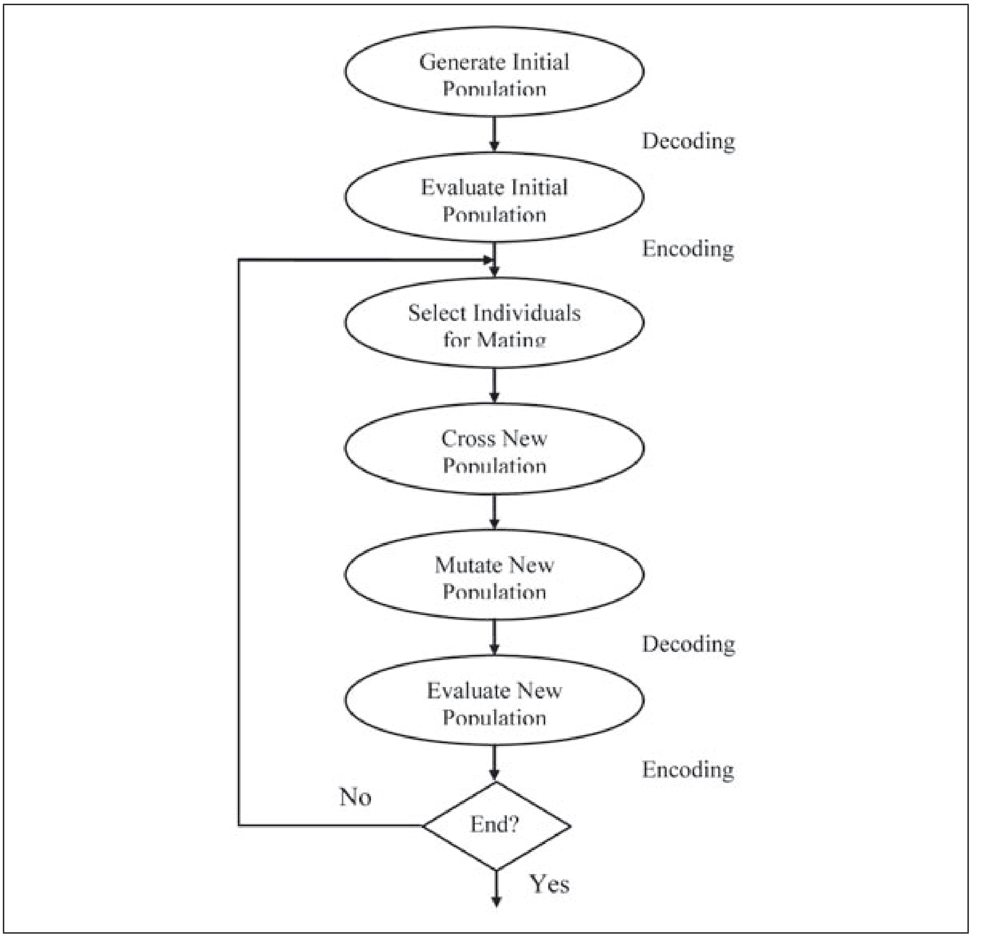
Selection determines which solutions are chosen for mating according to the principal of survival of the fittest (i.e. the better the performance of the solution, the more likely it is to be chosen for mating and therefore the more offspring it produces). In this work we used tournament selection. The tournament selection method works by choosing a group of q individuals randomly from the population and selecting the best individual in terms of fitness from this group.

Crossover allows an improvement in the species in terms of the evolution of new solutions that are fitter than any seen before. The crossover operator combines the features of two parents to create new solutions. One or several crossover points are selected at random on each parent and then, complementary fractions from the two parents are spliced together to form a new chromosome, as shown in Figure 1.

Mutation reintroduces values that might have been lost through selection or crossover, or creates totally new features. The mutation operator alters a copy of a chromosome. One or more locations are selected on the chromosome and replaced with new randomly generated values. Mutation is used to help ensure that all areas of the search space remain reachable providing higher variation in the chromosomes of each population. It also allows the reintroduction of features that might have been lost during the selection procedure.

The cycle selection-crossover-mutation-evaluation is performed until a termination criterion is met (for instance, a predetermined number of generations) (see Figure 2).

FIGURE 2
FLOW CHART OF A GENETIC ALGORITHM



For multi-objective *GA* (Coello, 2006) the concept of fitness changes. The fitness of an individual it is not anymore how well it solves the problem, but it is based on the Pareto optimality concept. The fitness of an individual is a function of how many individuals it dominates and by how many individuals it is dominated. Thus, nondominated individuals have the highest possible fitness and the rest of individuals are ranked according to their dominance relations.

Other important concept that is often used in *GAs* is the concept of elitism. In an elitist selection technique the best individuals of the population are automatically selected to go to the next generation without undergoing crossover or mutation. In the context of multiobjective *GAs*, the use of a subpopulation (usually called archive) where the nondominated individuals are stored along the generations guarantees that nondominated solutions are not lost during the run and that a solution reported as nondominated is nondominated with respect to any other solution generated by our algorithm.

3.1. GA IMPLEMENTATION

The *GA* implementation used in this work is based on *ECJ* (1), a research evolutionary computation system in Java developed at George Mason University's Evolutionary Computation Laboratory *(ECLab)*. For the multi-objective aspect of the optimization the *SPEA*2 (Strength Pareto Evolutionary Algorithm 2) package of *ECJ* was used (Zitzler *et al.* 2001). *SPEA*2 is an improved version of SPEA which incorporates a fine-grained fitness assignment strategy, a density estimation technique and an enhanced archive truncation method. As most of the multi-objective evolutionary methods it keeps an *archive* where the non-dominated solutions are stored. The size of the archive is set by the user so that if the number of non-dominated solutions is bigger than the archive size the archive is truncated.

The algorithm works as follows:

- In Step 1 and 2 the archive, *A(g)*, where the non-dominated solutions are stored and the population, *P(g)*, are initialized. *A*(0*)* is an empty set and *P*(0*)* is initialized at random.
- In Step 3 the generation counter *g* is set to 1 and then the evolution loop starts.
- In Step 4 and 5 the individuals in the population and the archive are evaluated.
- According to this evaluation a new archive is created in Step 6 containing all the non-dominated individuals found in the union of the previous archive and the population.
- If the size of the resulting archive exceeds the archive size, in Step 7 the archive is truncated. This truncation method removes those individuals which are at the minimum distance of another individual. This way the characteristics of the non-dominated front are preserved and outer solutions are not lost.
- The termination criterion in step 8 stops the algorithm when the number of generations has been completed.
- In Step 9 tournament selection with replacement is performed in the archive set in order to fill the mating pool, *M(g)*.
- The new population, *P(g)*, is created in Step 10 by applying crossover and mutation to the mating pool.
- In Step 11 the generation counter is increased.

The control parameters of the *GA* used are quite standard. The *GA* is generational. It uses tournament selection with tournament size of 7. The probabilities of crossover and mutation are 1 and 0.05 respectively. The population size is 5000 and the archive size is 100. The run finishes after 100 generations. Each individual is encoded as a vector of integers ranging from 0 to 99. Every element of the vector represents the percentage of the budget invested in that particular asset *(*$w_j^{GA} \leq 0$ $j = 1,\ldots, n$*)*. Therefore, the length of the vector equals the number of assets available in the portfolio. However, the summation of these

(1) http://cs.gmu.edu/~eclab/projects/ecj/

weights will not be 1, violating the constraint $\sum_{j=1}^{n} w_j = 1$. This constraint imposes the need of normalizing the vector during the decoding process as follows:

$$w_j = \frac{w_j^{GA}}{\sum_{j=1}^{n} w_j^{GA}} \tag{12}$$

where w_j represents the weight invested in asset *j*. However, these normalized weights are real values.

4. DATA AND EMPIRICAL ANALYSIS

The data used in this work were extracted from the Bloomberg database. It is a set composed of fifty stocks which belonged to the Eurostoxx 50 index in January 2008. Three stocks with negative expected return in the analysis period were eliminated. We use daily data of these stocks from January 2003 to December 2007. This gives us 1300 observations per stock. We chose daily data instead of monthly data to avoid inaccurate *VaR* estimates from small samples.

TABLE 1
SUMMARY OF DATA STATISTICS

Country	*Company*	*Mean*	*SD*	$VaR_{95\%}$	$CVaR_{95\%}$	*BJ*
France	Air Liquide	0.048	1.270	1.928	2.800	175.6
France	Alcatel-Lucent	0.004	2.333	3.472	5.457	640.5
Germany	Allianz, S.E.	0.039	1.853	2.911	4.398	985.6
Italy	Assicurazioni Generali SpA	0.041	1.243	1.771	2.832	308.1
France	AXA, S.A.	0.054	1.861	2.898	4.238	344.2
Spain	BBVA, S.A.	0.043	1.349	2.141	3.126	213.9
Spain	Banco Santander, S.A.	0.059	1.360	2.188	3.202	300.6
Germany	BASF, S.E.	0.074	1.429	2.061	3.087	623.4
Germany	Bayer, AG	0.087	1.993	2.551	4.161	3557.7
France	BNP Paribas	0.047	1.487	2.371	3.260	425.4
France	Carrefour, S.A.	0.015	1.432	2.171	3.254	1700.9
France	Cie de Saint-Gobain	0.060	1.608	2.580	3.629	2694.3
France	Credit Agricole, S.A.	0.036	1.485	2.210	3.322	112.9
Germany	Daimler. AG	0.056	1.650	2.717	3.630	24340.5
Germany	Deutsche Bank, AG	0.048	1.559	2.494	3.415	1227.9
Germany	Deutsche Boerse, AG	0.151	1.669	2.434	3.577	2190.6

(Continúa pág. sig.)

TABLE 1 (cont.)
SUMMARY OF DATA STATISTICS

Country	*Company*	*Mean*	*SD*	$VaR_{95\%}$	$CVaR_{95\%}$	*BJ*
Germany	Deutsche Telekom, AG	0.011	1.465	2.063	3.529	20488.1
Germany	E.ON, AG	0.099	1.418	2.074	3.163	6325.7
Italy	Enel, SpA	0.042	0.973	1.449	2.237	355.6
Italy	ENI, SpA	0.036	1.147	1.855	2.679	918.8
Belgium	Fortis	0.016	1.770	2.634	4.339	678.4
France	France Telecom, S.A.	0.036	1.641	2.339	3.621	238.27
France	Groupe Danone	0.049	1.222	1.705	2.470	1191.2
Spain	Iberdrola, S.A.	0.087	1.094	1.523	2.277	3887.1
Holland	ING Groep NV	0.035	1.851	2.827	4.389	8069.3
Italy	Intesa Sanpaolo, SpA	0.070	1.535	2.191	3.323	147.1
Holland	Koninklijke Philips Electronics, NV	0.037	1.828	2.800	4.075	548.9
France	L'Oreal, S.A.	0.021	1.347	2.021	3.032	871.2
France	LVMH, S.A.	0.054	1.369	2.166	2.984	1032.8
Germany	Muenchener Rueckversicherungs AG	0.011	1.845	2.738	4.533	2398.5
Finland	Nokia, OYJ	0.038	2.019	2.887	4.755	170.7
France	Renault, S.A.	0.056	1.721	2.745	3.842	459.7
Spain	Repsol YPF, S.A.	0.048	1.230	1.923	2.857	435.1
Germany	RWE, AG	0.100	1.535	2.273	3.456	944.7
France	Sanofi-Aventis, S.A.	0.005	1.471	2.235	3.426	7508.2
Germany	SAP, AG	0.043	1.719	2.430	3.798	539.9
Germany	Schneider Electric, SA	0.054	1.442	2.258	3.241	59.29
Germany	Siemens, AG	0.069	1.617	2.535	3.569	629.3
France	Societe Generale	0.042	1.550	2.416	3.537	6005.0
France	Suez, SA	0.076	1.952	2.657	4.582	668.3
Spain	Telefonica, SA	0.077	1.198	1.853	2.664	2330.4
France	Total, SA	0.038	1.240	2.143	2.774	339.6
Italy	UniCredit SpA	0.029	1.273	2.007	3.033	3106.8
Holland	Unilever NV	0.019	1.227	1.776	2.942	399.8
France	Vinci, SA	0.103	1.404	1.947	2.919	555.3
France	Vivendi	0.050	1.639	2.439	3.792	2778.3
Germany	Volkswagen, AG	0.107	1.773	2.717	3.789	200813.2

Mean: the average daily return. *SD*: standard deviation. $VaR_{95\%}$: the 95% 1-day *VaR*, and $CVaR_{95\%}$: the 95% 1-day *CVaR* are expressed in percentage and are computed with daily returns from January 2003 to December 2007. BJ is the Bera-Jarque statistic.

Table 1 reports the descriptive analysis of the data identifying companies by country. It can be observed that the mean daily return is close to zero. This is consistent with computing VaR under the assumption of expected daily return equal to zero. Table 1 reports standard deviation, *VaR* and *CVaR* at 95% confidence level for each stock. In order to compute the three risk measures we used historical simulation as mentioned previously (see Section 2.1).

Additionally, we compute BeraJarque *(BJ)* statistic to test normality. The *BJ* statistic has a chi-squared distribution with two degrees of freedom under the null hypothesis that returns are normally distributed. As can be observed the minimum value of the *BJ* statistic is 59.29 while the critical value is 9.21 for a 1% significance level. Hence normality is rejected in all cases and portfolio risk level ranking is different depending on the measure selected: standard deviation, *VaR* or *CVaR*. Also, Table 1 reports standard deviation, *VaR* and *CVaR* at 95% confidence level. For example, we can find Siemens AG with 1.617% of deviation, 2.535% of *VaR* and 3.569% of *CVaR* while *SAP AG* has higher deviation 1.719%, lower *VaR* 2.43% and higher *CVaR* 3.798%. These are only examples since they are not representative of the subset of efficient portfolios.

Although it is no trivial compare risk and return of efficient portfolios since the concept of efficiency is respect to a determine risk measure we show, through figures and tables, the differences between portfolios obtaining subsets of efficient portfolios under different risk measures (Model 1, 2 and 3) and translating them to different spaces. Furthermore, we previously evaluate the quality of the solution, subset of efficient portfolios, obtained with multiobjective genetic algorithm by comparing it with the solution obtained with an exact algorithm (Model 1A *versus* 1B, Model 3A *versus* 3B).

For testing the reliability of the multi-objective evolutionary approach *(GA)*, we solved the classical mean-variance problem using *GAs*, Model 1B, and we compared the results obtained with those of the classical Quadratic Program *(QP)*, Model 1A. Also, we solved the mean-*CVaR* problem using *GAs*, Model 3B, and we compared the results obtained with those of linear program *(LP)* proposed by Rockefellar and Uryasev (2000, 2002), Model 3A.

TABLE 2

DIFFERENCES IN DEVIATION AND CVaR AMONG OPTIMAL PORTFOLIOS CALCULATED USING DIFFERENT OPTIMIZATION TECHNIQUES

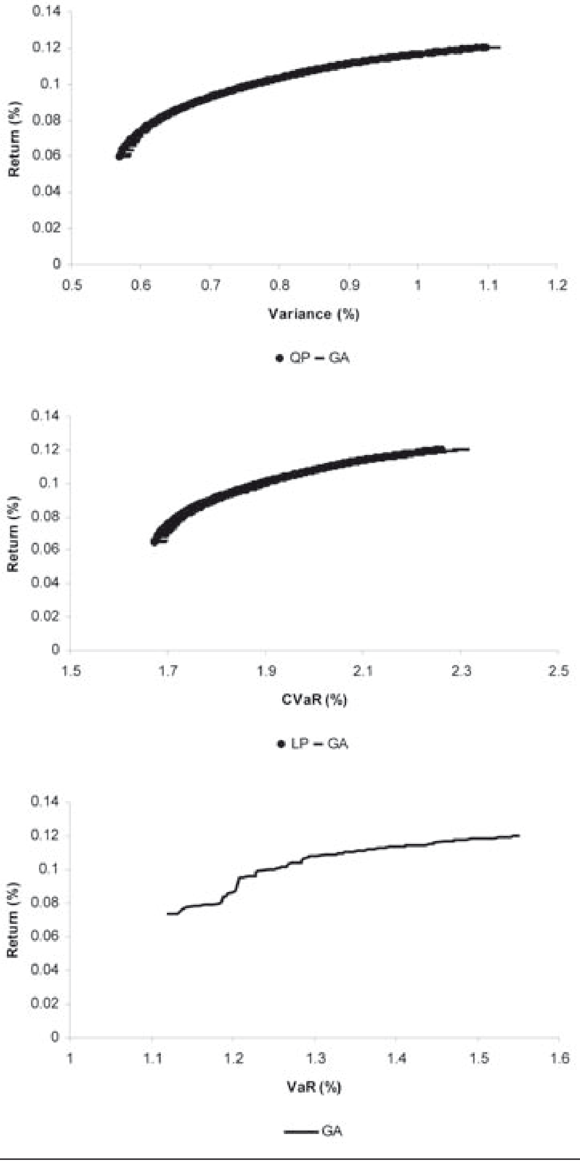
	$W_{\sigma}^{GA} - W_{\sigma}^{QP}$	$W_{CVaR}^{GA} - W_{CVaR}^{LP}$
Mean	0.0049	0.0190
Deviation	0.0027	0.0081
Min.	0.0014	0.0092
Max.	0.0196	0.0430

NOTE: values are expressed in percentage.

Figure 3 illustrates that the σ-efficient frontiers calculated using *GAs* and *QP* (Model 1*A* and 1*B*) and *CVaR*-efficient frontiers calculated using *GAs* and *LP* (Model 3*A* and 3*B*) are overlapped. Also, it is showed the *VaR*-efficient frontier obtained using *GAs* (Model 2). In this case, there is not an exact algorithm available and heuristic methods are essential.

As it is showed the efficient frontier irregularity reflects that *VaR* is not a convex risk measure.

FIGURE 3
MEAN-VARIANCE, MEAN-*CVaR* AND MEAN-*VaR*
EFFICIENT FRONTIERS COMPUTED WITH DIFFERENT TECHNIQUES



Mean-variance efficient frontier was computed using multiobjective *GA* and quadratic programming, *QP*. Mean-*CVaR* efficient frontier was computed using multiobjective *GA* and linear programming, *LP*. Mean-*VaR* efficient frontier was computed using multiobjective *GA*.

Table 2 reports the differences between efficient frontiers. For each level of return, $w_{\sigma}^{GA} - w_{\sigma}^{QP}$ is the difference in standard deviation between a portfolio obtained with *GA* (Model 1*B*) and *QP* (Model 1*A*). The mean error of the σ-efficient portfolios computed using *QP* (Model 1*A*) and *GA* (Model 1*B*) is close to zero (0.0049%). The efficient portfolio with the highest difference is 0.0196% which further confirms the good results provided by the multiobjective genetic algorithm to minimize deviation and maximize return. Similar conclusions are obtained when we maximize return and minimize *CVaR* (Model 3*B*) and we compare these results of the linear programming applying the transformation of the objective function proposed by Rockefellar and Uryasev (2000, 2002), (Model 3*A*). For each level of return, $w_{CVAR}^{GA} - w_{CVAR}^{LP}$ is the difference in *CVaR* value between a portfolio obtained with multiobjective *GA* (Model 3*B*) and *LP* (Model 3*A*). The mean error of the *CVaR*-efficient portfolios computed using *LP* and *GA* is close to zero (0.0190%), the efficient portfolio with the highest difference is 0.0430%.

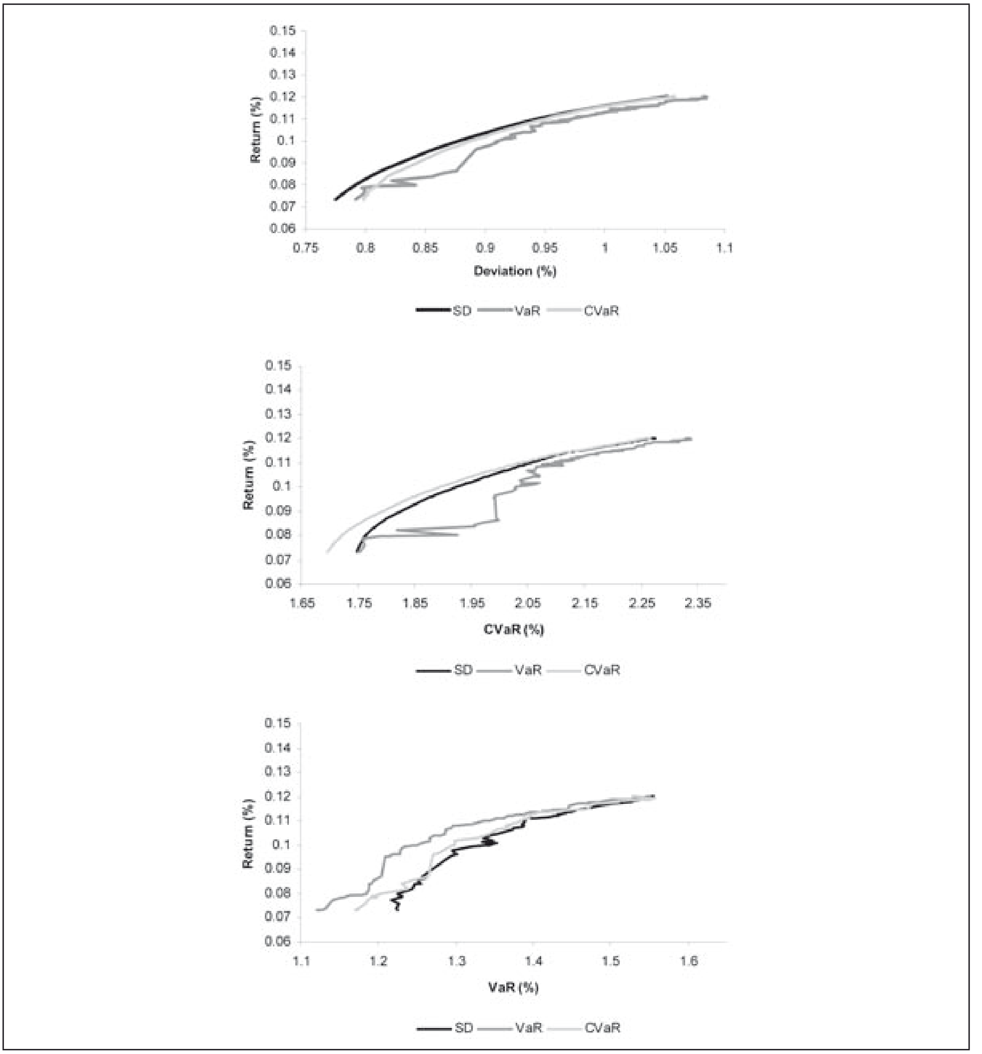
So far we have proven that multiobjective *GAs* can easily solve Markowitz classical problem and the mean-*CVaR* problem without linear transformation. We want to quantify how different these risk measures (deviation, *CVaR* and *VaR*) are in the subset of efficient portfolios to know if this risk measures are interchangeable. Of course, as investors choose a portfolio in the subset of efficient portfolios to know if a risk measure is interchangeable in the set of feasible portfolios is irrelevant

To deal with the task of comparing efficient frontier under different risk measures, we use the efficient portfolios obtained under the same algorithm (Model 1*B*, 2 and 3*B*). For each efficient portfolio under a determined risk measure we compute the portfolio risk applying the other two definitions of risk not used to determine that it was an efficient portfolio. For example, for each efficient portfolio obtained from mean-variance model (Model 1*B*), named σ-efficient portfolio, we compute its *VaR* and *CVaR* in order to compare them with the efficient portfolios obtained from mean-*VaR* Model (Model 2), named *VaR*-efficient portfolios, and with the efficient portfolios obtained from mean-*CVaR* model (Model 3*B*), named *CVaR*-efficient portfolios.

When investors use Model 1, 2 or 3, they choose an efficient portfolio under one risk measure and ignore the other measures of risk. It is not wrong when we are assuming that the used risk measure and the ignored are substitutive, that is, all the risk measures give similar efficient frontier. Then the question is: deviation of σ-efficient portfolios (Model 1*B*), deviation of *VaR*-efficient portfolios (Model 2) and deviation of *CVaR*-efficient portfolios (Model 3*B*) are equivalent? Alternatively, *VaR* of σ-efficient portfolios (Model 1*B*), *VaR* of *VaR*-efficient portfolios (Model 2) and *VaR* of *CVaR*-efficient portfolios (Model 3*B*) are equivalent?. And finally, *CVaR* of σ-efficient portfolios (Model 1*B*), *CVaR* of *VaR*-efficient portfolios (Model 2) and *CVaR* of *CVaR*-efficient portfolios (Model 3*B*) are equivalent? We only know that deviation of σ-efficient portfolios (Model 1*B*), *VaR* of *VaR*-efficient portfolios (Model 2) and *CVaR* of *CVaR*-efficient portfolios (Model 3*B*) are the lowest. Graphically, it can be answered representing the portfolios in different spaces, (mean-variance space, mean-*VaR* space and mean-*VaR* space), Figure 4. If the three risk measures are substituted, then the efficient frontiers should be overlapped in the three spaces and investors could consider whatever one in the portfolio selection model choosing between mean-variance, mean-*VaR* or mean-*CVaR* model (Model 1, 2 or 3). Of course the investors' election could be the simplest or better-known portfolio selection model (mean-variance model).

On the other hand, if the three risk measures are complementary, then the efficient frontiers are not overlapped in the three spaces. In this case, it is worthwhile to include more than one risk measure in the objective function. Investors should define their preferences in terms of the different risk measures and select a portfolio from the mean-variance-*VaR*-*CVaR* model since all the risk measures are jointly considered for obtaining the efficient portfolios.

FIGURE 4
MEAN-VARIANCE, MEAN-*CVAR* AND MEAN-*VAR* EFFICIENT FRONTIERS REPRESENTED IN MEAN-VARIANCE, MEAN-*CVAR* AND MEAN-*VAR* SPACES



SD line represents σ-optimal porfolios, *VaR* line reprents *VaR*-optimal porfolios and *CVaR* line represents *CVaR*-optimal portfolios.

TABLE 3

DIFFERENCES IN DEVIATION AND *CVaR* AMONG OPTIMAL PORTFOLIOS UNDER DIFFERENT RISK MEASURES

	SD differences		*CvaR differences*	
	$w_\sigma - w_{CVaR}$	$w_\sigma - w_{VaR}$	$w_{CVaR} - w_\sigma$	$w_{CVaR} - w_{VaR}$
Mean	0.0080	0.0314	0.0201	0.0895
Deviation	0.0055	0.0095	0.0136	0.0443
Min	0.0020	0.0085	0.0038	0.0411
Max	0.0237	0.0630	0.0531	0.2362

NOTE: Differences in deviation and *CvaR* are expressed in percentage. w_σ, w_{CVaR} and w_{VaR}– represent optimal portfolios under deviation, *CVaR* and *VaR* risk measures, respectively.

Figure 4 illustrates differences between σ-efficient portfolios, *CVaR*-efficient portfolios and *VaR*-efficient portfolios in mean-deviation, mean-*CVaR* and mean-*VaR* spaces. As can be observed, even though in each space the efficient portfolios are those obtained as optimal for the risk measure that define the space, σ-efficient portfolios and *CVaR*-efficient portfolios are close in all spaces while *VaR*-efficient portfolios are more separated in all spaces. This confirms that obtaining efficient portfolios for several risk measures is not always so important. Concretely, when the risk measures considered are standard deviation and *CVaR* the efficient portfolios are similar. However, when *VaR* is considered as risk measure, the inclusion of other risk measure such as *CVaR* or variance gives extra information to the investor.

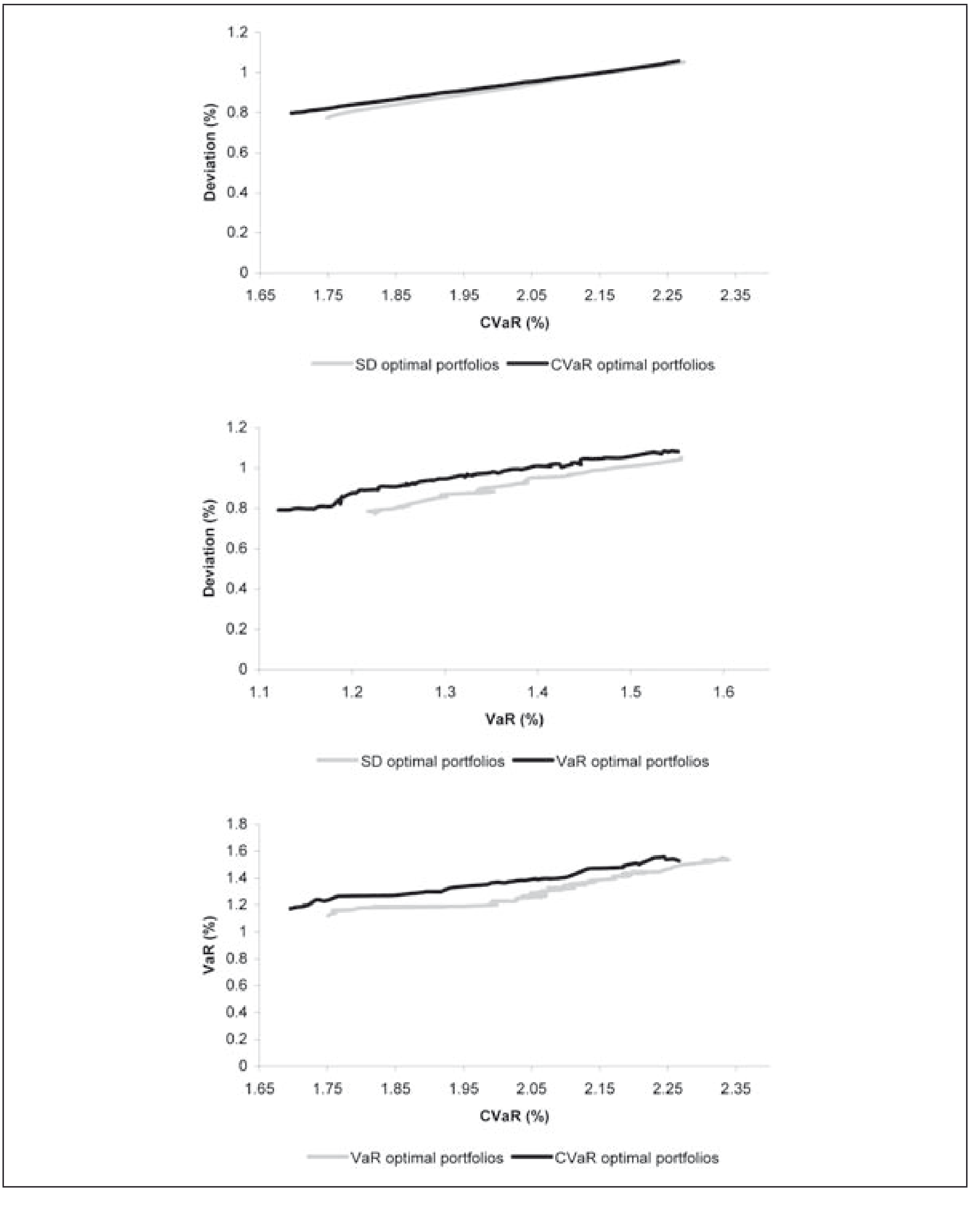
Table 3 quantifies the difference in deviation and *CVaR* between efficient portfolios under different risk measures. The mean difference in terms of portfolio standard deviation between σ-efficient portfolios, named w_σ, (Model 1*B*) and *CVaR*-efficient portfolios, named w_{CVaR}, (Model 3*B*) is 0.008% with a maximum difference of 0.0237% while the mean difference with VaR-efficient portfolios, named w_{VaR}, (Model 2) is 0.0314% with a maximum difference of 0.063%. This means that deviation is a measure more related to *CVaR* than *VaR*. With respect to *CVaR*, the mean difference between the *CVaR* of *CVaR*-efficient portfolios (Model 3*B*) and the *CVaR* of σ-efficient portfolios (Model 1*B*) is 0.0201% with a maximum difference of 0.0531% while the mean difference with *VaR*-efficient portfolios (Model 2) is 0.0895% with a maximum difference of 0.2362%.

Figure 5 presents the efficient portfolios: σ-efficient portfolios (Model 1*B*), *CVaR*-efficient portfolios (Model 3*B*) and *VaR*-efficient portfolios (Model 2) are represented in the σ-*CVaR*, σ-*VaR* and *VaR*-*CVaR* spaces for different fixed returns. The differences between σ-efficient portfolios and *CVaR*-efficient portfolios in the σ-*CVaR* space are the smallest and the differences between *VaR*-efficient portfolios and *CVaR*-efficient portfolios in the *VaR*-*CVaR* space are the biggest as Table 3 reflects.

These results reflect a more similarity between σ-efficient portfolios (Model 1*B*) and *CVaR*-efficient portfolios (Model 3*B*) than between them and *VaR*-efficient portfolios (Model 2). These results indicate that it could be interesting for investors to incorporate *VaR* and other risk measure, variance or *CVaR*, in the portfolio optimization problem. Investors could choose between solving the mean-*VaR*-*CVaR* model (Model 4) or the mean-variance-*VaR* model

(Model 5). These models 4 and 5 are multiobjective as models 1*B*, 2 and 3*B*, and they can not be reduced to one objective function as it is done in model 1*A* and 3*A* by solving them for different values of *r**. But they can be solved easily applying a multiobjective genetic algorithm.

FIGURE 5
σ-OPTIMAL PORTFOLIOS, *CVaR*-OPTIMAL PORTFOLIOS AND *VaR*-OPTIMAL PORTFOLIOS REPRESENTED IN THE σ-*CVaR*, σ-*VaR* AND *VaR*-*CVaR* SPACES



It has to be pointed out that as σ-efficient portfolios are the efficient portfolios in mean-variance space they are also efficient in mean-variance-*VaR* or mean-variance-*CVaR* space. This is due to the fact that for a fixed return their volatility is the minimum. That is, other portfolios could have a smaller *CVaR* or *VaR* for the same return but not a smaller deviation. The same happens with *CVaR*-efficient portfolios. They are also efficient or non-dominated in mean-variance-*CVaR* space. Therefore, mean-variance-*CVaR* efficient portfolios are all the portfolios included in the area between σ-efficient portfolios and *CVaR*-efficient portfolios. As is shown in Figure 5 this surface is narrow, which means that the set of portfolios in which investors could select their portfolio selection is similar solving mean-variance model (Model 1*A* or 1*B*), mean-*CVaR* model (Model 3*A* or 3*B*) or mean-variance-*CVaR* model presented in Roman *et al.* (2007). However, mean-*VaR*-*CVaR* and mean-variance-*VaR* model (Model 4 and 5) offer wider surface of efficient portfolios to investors. Therefore, it is interesting to solve them since the solutions obtained offer a wide range of portfolios for the investors to choose according to their preferences. Similarly, taking into consideration the asset allocation with the three risk measures, that is, a mean-variance-*VaR*-*CVaR* model, would be similar to consider mean-*VaR*-*CVaR* or mean-variance-*VaR* models since variance and *CVaR* in the set of efficient portfolios generate a narrow surface.

TABLE 4

RISK ATTRIBUTES OF EFFICIENT PORTFOLIOS IN SEVERAL SPACES

	r = 0.08%			r = 0.10%		
	SD	*CVaR*	*VaR*	*SD*	*CVaR*	*VaR*
Min σ	0.791*	1.764	1.225	0.877*	1.923	1.343
Min *CVaR*	0.810	1.723*	1.215	0.887	1.894*	1.305
Min *VaR*	0.841	1.926	1.185*	0.911	2.028	1.251*
Min *VaR*, σ	0.805	1.784	1.208	0.898	1.987	1.303
Min *VaR,CVaR*	0.808	1.770	1.203	0.907	1.948	1.295

* Indicates minimum value for given return.

Table 4 illustrates the risk values of the σ-efficient portfolios (Model 1*B*), the *CVaR*-efficient portfolios (Model 3*B*), the *VaR*-efficient portfolios (Model 2), the σ-*VaR*-efficient portfolios (Model 5) and the *VaR*-*CVaR*-efficient portfolios (Model 4) for two level of expected return, 0.08% and 0.1%. Minimum values of σ, *VaR* and *CVaR* values are highlighted for each return. As can be observed, for each return σ-*VaR*-efficient portfolios have a higher deviation and lower *VaR* than σ-efficient portfolios and lower deviation and higher *VaR* than *VaR*-efficient portfolios. σ-*VaR*-efficient portfolios balance deviation and *VaR* and graphically all the portfolios that make up the efficient surface are located in the gap shown in Figure 5 when σ-efficient portfolios (Model 1B) and *VaR*-efficient portfolios (Model 2) are represented in σ-*VaR* space. The same happens when we analyze the *VaR*-*CVaR*-efficient portfolios for each level of return proposed. They have a lower *VaR* and higher *CVaR* than *CVaR*-efficient portfolios and higher *VaR* and lower *CVaR* than *VaR*-efficient portfolios. The surface of efficient portfolios for mean-*VaR*-*CVaR* model is composed by portfolios in the gap between *VaR*-efficient portfolios and *CVaR*-efficient portfolios represented in the *VaR*-*CVaR* space in Figure 5.

5. CONCLUSIONS

Variance, *VaR* and *CVaR* are different risk measures that capture different aspects of the return distribution related to risk. Usually asset allocation has been done through a multiobjective problem where expected return is maximized and one risk measure is minimized. To reduce one objective function, the problem is solved for proposed values of expected return. The inclusion of several risk measures is an issue that is being addressed recently by researchers. In this paper, we evaluate which combination of risk measures is more informative. In doing this, if the portfolio return is a normal distribution, variance, *VaR* and *CVaR*, generate the same efficient frontier since they are proportional. We used a historical simulation method to compute these risk measures in order to allow nonlinearities and avoid constraining conclusions to normal distribution assumption.

Investors decide about the risk measures considered to choose the investment portfolio. Based on our results we can conclude that investors do not need balance variance and *CVaR* when selecting an investment portfolio since efficient portfolios in the variance-*CVaR* space form a narrow surface. Then, ignoring a risk dimension does not change portfolio risk values significantly. If investor decide not ignoring them, after fixing a required return the range of variance and *CVaR* values to choose is too small. However, VaR efficient portfolios have a different level of variance and *CVaR* than variance efficient portfolios and *CVaR* efficient portfolios. As a result, we show that in the space variance-*VaR* and *VaR-CVaR* these portfolios generate a wider surface to balance investors' preferences based on two risk measures.

Consequently, it is more informative to solve a mean-variance-*VaR* model or a mean-*VaR-CVaR* model since variance and *VaR* or *VaR* and *CVaR* behave differently and, as a consequence the resulting surface of efficient portfolios is significantly wider. Investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is *VaR*, for an expected return required and variance level ranging between the variance of the mean-variance efficient portfolios and the mean-*VaR* efficient portfolios. Alternatively and with similar results, investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is *VaR*, for an expected return required and mean loss under the *VaR* level ranging between the *CVaR* of the mean-*CVaR* efficient portfolio and the mean-*VaR* efficient portfolio. In a general framework, where normality is not assumed, it is recommendable combining *VaR* with variance or *CVaR* in order to select an investment portfolio.

Finally, the results obtained show the feasibility of using multiobjective genetic algorithms for portfolio optimization problems under general risk definitions and several risk measures. The multiobjective genetic algorithm deals with the non-convexity characteristics of *VaR* and it solves the three-objectives models efficiently.

REFERENCES

ALEXANDER, G. J.; BAPTISTA, A. M., and YAN, S. 2007. Mean-variance portfolio selection with 'at-risk' constrains and discrete distributions. *Journal of Banking and Finance* 31: 3761-3781.

ARTZNER, P.; DELBAEN, F.; ERBER, JM, and HEATH, D. 1999. Coherent measures of risk. *Mathematical Finance* 9: 203-228.

BAIXAULI, J. S., and ALVAREZ, S. 2004. Analysis of the conditional stock-return distribution under incomplete specification. *European Journal of Operational Research* 155: 276-283.

BAIXAULI-SOLER, J. S.; ALFARO-CID, E., and FERNANDEZ-BLANCO, M. O. 2010. Mean-VaR portfolio selection under real constraints. *Computational Economics*. DOI: 10.1007/s10614-009-9195-1.

BASEL COMMITTEE ON BANKING SUPERVISION. 2001. *The New Basel Capital Accord.* Available in http://www.bis.org/ (6 junio 2010)

BAWA, V. S. 1975. Optimal rules for ordering uncertain prospects. *Journal of Financial Economics* 2: 95-121.

COELLO, C. A. 2006. Evolutionary multi-objective optimization: A historical view of the field. *IEEE Computational Intelligence Magazine* 1: 28-36.

DUFFIE, D., and PAN, J. 1997. An overview of value-at-risk. *Journal of Derivatives* 4: 7-49.

FISBURN, P. C. 1977. Mean-risk analysis with risk associated with below-target returns. *American Economic Review* 67: 116-126.

GAIVORONSKI, A. A. and PFLUG, G. 2005. Value-at-risk in portfolio optimization: properties and computational approach. *Journal of Risk* 7: 1-31.

GILLI, M.; KËLLEZI, E., and HYSI, H. 2006. A data-driven optimization heuristic for downside risk minimization. *Journal of Risk* 8: 1-18.

GOLDBERG, D. 1989. *Genetic Algorithms in Searching Optimization and Machine Learning.* Addison-Wesley, Reading, MA.

HOLLAND, J. H. 1975. *Adaptation in Natural and Artificial Systems.* University of Michigan Press. Ann Arbor.

JORION, P. 1996. *Value at risk: a new benchmark for measuring derivatives risk.* Irwin professional publishing.

KONNO, H.; SHIRAKAWA, H., and Yamazaki, H. 1993. A mean-absolute deviation-skewness portfolio optimisation model. *Annals of Operational Research* 45: 205-220.

KONNO, H., and SUZUKI, K. 1995. A-mean-variance-skewness portfolio optimisation model. *Journal of Operations Research Society of Japan* 38: 173-187.

LIM, C.; SHERALI, H. D., and URYASEV, S. 2010. Portfolio optimization by minimizing conditional value-at-risk via nondifferentiable optimization. *Computational Optimization and Applications*. 46: 391-415.

MARKOWITZ, H. 1952. Portfolio selection. *Journal of Finance* 7: 7-91.

ROCKAFELLAR, R. T., and URYASEV, S. 2000. Optimization of conditional value-at-risk. *Journal of Risk* 2: 21-41.

ROCKAFELLAR, R. T., and URYASEV, S. 2002. Conditional value-at-risk for general loss distributions. *Journal of Banking and Finance* 26: 1443-1471.

ROMAN, D.; DARBY-DOWMAN, K., and MITRA, G. 2007. Mean-risk models using two risk measures: a multi-objective approach. *Quantitative Finance* 7: 443-458.

ZITZLER, E.; LAUMANNS, M., and THIELE, L. 2001. *SPEA2: Improving the Strength Pareto Evolutionary Algorithm,* Technical Report 103. Swiss Federal Institute of Technology.