Several risk measures in portfolio selection: Is it worthwhile? *

¿Está justificado el uso de varias medidas de riesgo en la selección de carteras?

J. Samuel Baixauli-Soler**. Universidad de Murcia
Eva Alfaro-Cid. Instituto Tecnológico de Informática (Valencia)
Matilde O. Fernández-Blanco. Universidad de Valencia

ABSTRACT This paper is concerned with asset allocation using a set of three widely used risk measures, which are the variance or deviation, Value at Risk and the Conditional Value at Risk. Our purpose is to evaluate whether solving the asset allocation problem under several risk measures is worthwhile or not, given the added computational complexity. The main contribution of the paper is the solution of two models that consider several risk measures: the mean-variance-VaR model and the mean-VaR-CVaR model. The inclusion of VaR as one of the objectives to minimize leads to nonconvex problems, therefore the approach we propose is based on a heuristic: multi-objective genetic algorithms. Our results show the adequacy of the multi-objective approach for the portfolio optimization problem and emphasize the importance of dealing with mean-σ-VaR or mean-VaR-CVaR models as opposed to mean-σ-CVaR, where both risk measures are redundant.

KEYWORDS Portfolio Selection; Value-at-Risk; Conditional Value-at-Risk; Multi-Objective Decision Making.

1. INTRODUCTION

Portfolio selection is obtained maximizing expected return and minimizing risk. There are several ways of measure risk of a portfolio. The classical measure of risk is the variance or

* Acknowledgment: We would like to thank the financial support from the Spanish Ministry of Science and Innovation Project ECO2008-02486.
** Corresponding author: J. Samuel Baixauli. Departamento de Organización de Empresas y Finanzas. Universidad de Murcia, Campus de Espinardo, s/n. 80100 Murcia. Correo electrónico: sbaixauli@um.es.
standard deviation used by seminal work of Markowitz (1952). In this case, the portfolio selection is done by solving a quadratic problem. This risk measure weights equally positive against negative returns. This is justified by assuming either that investors have quadratic utility functions or that asset returns are drawn from a multivariate elliptical distribution. Other measures known as downside risk measures have been proposed to capture the left-hand side of a return distribution which involves risks. Bawa (1975) and Fishburn (1977) introduce a general definition of downside risk in the form of lower partial moments. An example is semi-variance, which is a special case of lower partial moments.

Nowadays researchers and practitioners are focused on Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) as measures of market risk. VaR of a portfolio is the lowest amount such that the loss will not exceed it with probability 1-α. CVaR is the conditional expectation of losses above the VaR. VaR and CVaR can be used to balance risk and return. While CVaR can be efficiently minimized using linear programming and non-smoothing techniques (Rockafellar and Uryasev, 2000), minimizing VaR leads to a non-convex and non-differential risk-return problem and smoothing techniques (Gaivoronski and Pflug, 2005) or heuristic optimization techniques need to be applied (Gilli et al., 2006). Although variance is still the most widely used measure of risk in the practice of portfolio selection, VaR and CVaR are used as risk limit and to control risk by the fund management industry. This has motivated the inclusion of VaR and CVaR as constraints in the classical mean-variance problem (Alexander et al., 2007).

Each risk measure, variance, VaR and CVaR, captures different aspects of risk. Therefore, it could be worthy to introduce them jointly in the portfolio analysis. In the literature we can find some works that explore this idea. For instance, Roman et al., (2007) proposed the use of two risk measures (variance and CVaR) in order to perform a portfolio selection. The mean-variance-CVaR model proposed is a multi-objective problem. A method to obtain all the efficient solutions is given and tested on the solutions of mean-variance model and mean-CVaR model.

This paper contributes to the literature in several ways. Firstly, it empirically shows the validity of multiobjective genetic algorithms for obtaining efficient portfolios under different risk measures including VaR. Secondly, even though normality is not assumed, we show that variance and CVaR identify similar efficient portfolios and thirdly, we show how multiobjective genetic algorithms can deal with three explicit objective functions: maximize return and minimize two risk measures without making assumptions about the asset return distribution. The proposal is validated on real data from Eurostoxx 50 index.

The structure of the paper is as follow: Section 2 formulates the problems under one and several risk measures. Section 3 presents the multiobjective genetic algorithm proposed. Data and empirical results are presented in Section 4. Finally, in Section 5 some conclusions are drawn.

2. THE PORTFOLIO SELECTION PROBLEM UNDER A SET OF RISK MEASURES

Although investors may propose different risk measures, the three stages of Markowitz framework are still valid: the stage of security analysis, which comprises the collection of
data and the estimation of relevant parameters; the stage of portfolio analysis, where the efficient set of portfolios is computed by solving an optimization problem dependant on the risk measure; and the stage of portfolio choice, which entails the selection of an optimal portfolio depending on the specific preferences of the investor.

2.1. Risk Measures

The choice of a measure of risk plays a fundamental role for the portfolio optimization problem. Following Artzner et al. (1999), a risk measure can be viewed as a single number $\rho(r)$ assigned to the distribution of the portfolio return $r$. This risk measure should satisfy four properties: monotonicity, translation invariance, homogeneity and subadditivity.

Monotonicity: if $r_1 \leq r_2$, then $\rho(r_1) \leq \rho(r_2)$, that is, if a portfolio has systematically lower returns than another for all states of the world, its risk must be greater.

Translation invariance: $\rho(r + k) = \rho(r) - k$, that is, adding cash $k$ to a portfolio should reduce its risk by $k$.

Homogeneity: $\rho(br) = b\rho(r)$, that is, increasing the size of a portfolio by $b$ should scale its risk by the same factor.

Subadditivity: $\rho(r_1 + r_2) \leq \rho(r_1) + \rho(r_2)$, that is, merging portfolios cannot increase risk.

Unfortunately, there are several risk measures that satisfy these properties. We can identify, at least, three widely used risk measures: variance, VaR and CVaR. Variance and CVaR satisfy the four properties and VaR satisfies three of them. VaR does not satisfy subadditivity under certain conditions.

2.1.1. Variance

Variance or standard deviation of returns is a traditional risk measure which measures volatility. In Markowitz formulation risk is defined as the variance of returns, that is the weighted sum of squared deviations around the mean. Therefore risk is measured as the dispersion of possible outcomes. A flatter distribution indicates greater risk, and a tighter distribution, lower risk.

If $R_{j} = 1, \ldots, T$ are historical observations of $r$, the expected return is estimated by the sample mean, $\bar{R} = \frac{1}{T} \sum_{j=1}^{T} R_{j}$, the variance is estimated by the sample variance, $\sigma^2 = \frac{1}{T} \sum_{j=1}^{T} (R_{j} - \bar{R})^2$, and the squared root of $\sigma^2$ is the standard deviation of rates of return. Variance treats symmetrically positive or negative deviations from the mean and, when returns are asymmetric, it can give a partial picture of risk.

2.1.2. VaR

The quantity identifies, with a certain confidence interval, the maximum anticipated loss in portfolio due to adverse market movements. It is a compact representation of risk level and
measures downside risk. By definition VaR is a quantile of the probability distribution of the portfolio value. Let \( f(r) \) the probability distribution function of the future portfolio return and \( \alpha \) the significant level (usually 5\%), \( \text{VaR} \) is implicitly defined in the following equation, \[ \int_{-\infty}^{\text{VaR}} f(r)dr = \alpha, \] and using the cumulative distribution function \( F(r) \) can be explicitly defined as, \( \text{VaR}_\alpha = \inf \{ r | F(r) \geq \alpha \} \).

The \( \text{VaR} \) computation can be simplified considerably if the distribution \( f(r) \) can be assumed to belong to a parametric family, such as the normal distribution. In this case, the \( \text{VaR} \) can be derived directly from the portfolio standard deviation using a multiplicative factor that depends on the confidence level. For example, to find \( \text{VaR} \) for 5\% left-tail significant level of a normal variable corresponds to a -1.645 times the standard deviation since -1.645 is the quantile of the standard normal distribution. Hence, \( \text{VaR} \) is simply a multiple of the standard deviation which is related to the confidence level. Under this assumption, \( \text{VaR} \) and variance are interchangeable so independently of using \( \text{VaR} \) or variance we obtain the same efficient portfolio. The issue is which distribution may fit the data. It is not trivial, an approach to test the conditional distribution can be found in Baixauli and Alvarez (2004).

In this work we compute \( \text{VaR} \) using the historical simulation method, which is perhaps the most widely used method to compute \( \text{VaR} \). It consists of going back in time and applying current weights to a time-series of historical asset returns. By keeping weights at their current values the history of a hypothetical portfolio is reconstructed.

\[ R_{p,j} = \sum_{i=1}^{N} w_i R_{i,j} \quad \text{for} \quad j = 1,...,T \]  

\( \text{VaR} \) is obtained from the entire distribution of hypothetical returns, where each historical day is assigned the same probability of occurrence \((1/T)\). By sorting in ascending order \( R_{p,j} \) we choose the \( R_{p,j}^{*} \) in the position \( \alpha T \).

This method is simple to implement, it short-circuits the need for an estimation of a covariance matrix, it simplifies the computations and by relying on actual prices, it allows non-linearities and non-normal distributions. This method captures correlations and it does not rely on specific assumptions about valuation models or the underlying stochastic structure of the market. Perhaps most important, it can account for fat tails and asymmetries. Under the historical simulation model \( \text{VaR} \) and variance are not interchangeable and the efficient frontier could, and probably will, be different.

In Artzner et al. (1999) the \( \text{VaR} \) measure was criticized due to the fact that it does not always satisfy subadditivity, one of the four properties for a risk measure to be coherent. However, Basel Committee (2001) assumes VaR as a risk measure and the regulatory capital for a loan is correlated to its marginal contribution to VaR. Finally, \( \text{VaR} \) is a non-convex risk measure, which makes it difficult to use in optimization problems.
2.1.3. CVaR

CVaR, for continuous distributions also known as the mean excess loss, mean shortfall and tail VaR, is defined as the average of all losses exceeding the VaR and it is computed as the expected value of \( r \) conditional on exceeding the VaR,

\[
CVaR_{\alpha} = \frac{\int_{VaR}^{\infty} r f(r) dr}{\int_{VaR}^{\infty} f(r) dr}
\]

This risk measure informs about how much we could lose if the portfolio return falls beyond VaR. Rockafellar and Uryasev (2000) proposed CVaR as a risk measure which overcomes the problems posed by the use of VaR for capital allocation. CVaR satisfies the four properties and it is considered a coherent risk measure. However, it is not as popular as VaR since financial institutions have promoted VaR as standard measure of market risk.

Also, CVaR computation can be simplified if the distribution \( f(r) \) is assume to be normal. In this case, CVaR can be expressed as follows:

\[
CVaR_{\alpha} = \frac{\alpha VaR - \mu}{\sigma} - \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(VaR - \mu)^2}{2\sigma^2}}
\]

CVaR captures the shape of the tail of the distribution \( f(r) \) and when we assume normality we are restricting this shape to a functional form. For example, in a standard normal distribution, given the significant level of 5%, VaR is -1.645 and CVaR is -2.062. Again, under this assumption VaR, CVaR and variance are interchangeable in a portfolio optimization problem since the value of these risk measure for different portfolios gives the same ranking, that is, if a portfolio A has a variance greater than portfolio B then VaR of portfolio A will be greater than VaR of portfolio B and CVaR of portfolio A will be greater than CVaR of portfolio B. Therefore when a return distribution is assumed to be normal the mean-variance model, mean-CVaR model and mean-VaR model to solve for obtaining the efficient set of portfolios are equivalent.

In this work we compute CVaR using the historical simulation method as in the case of VaR, that is, we obtain \( CVaR \) from the entire distribution of historical returns, as the sample mean of \( R_{pj} \) lower than VaR

\[
CVaR_{\alpha} = \frac{\sum_{j=1}^{\alpha T} R_{pj}}{\alpha T}
\]

Obtaining CVaR by historical simulation method has the same advantages than using historical simulation for calculating VaR. It is simple to implement and it allows non-normal
distribution. Under the historical simulation model $CVaR$, $VaR$ and variance are not interchangeable and the efficient frontier could be different.

Variance and $CVaR$ are convex risk measures. The convexity of an objective function is an essential point for demonstrating the convergence of numerical optimization methods because for convex functions every local minimum is a global one and the set of optimal strategies is convex. However, $VaR$ is a non-convex risk measure and it makes it difficult to use in optimization problems.

In order to show the irregularity of $VaR$ against standard deviation and $CVaR$, Figure 1 from Baixauli et al. (2010) shows the $VaR$, standard deviation and $CVaR$ for all feasible portfolios composed of two stocks (Total, S.A. and Banco Santander) computed with historical simulation method. The $x$-axis shows the percentage of the budget invested in Total, S.A. while one minus this value is the percentage invested in Banco Santander. Figure 1 shows the convexity of the standard deviation and $CVaR$ with a global minimum whereas $VaR$ is not a convex function and it presents several local minimums. Moreover, the standard deviation, $VaR$ and $CVaR$ are not interchangeable risk measures because the minimum deviation portfolio and the minimum $VaR$ portfolio do not coincide. Total, S.A. weight in the optimal portfolio using deviation is 59%, using $VaR$ is 49% and using $CVaR$ is 68%.

**Figure 1**

DEVIATION, $VA$ AND $CVAR$ FOR PORTFOLIOS WITH TWO ASSETS

Portfolios are composed by Total, S.A. and Banco Santander, S.A.

Obviously, this is only an example of a portfolio composed by two securities but it illustrates the issue that we address in this paper with $n$ securities.
2.2. Portfolio selection under one risk measure

When a new risk measure, such as VaR and CVaR, is proposed, the computational complexity of the optimization problem can increase. Hence, the problem to solve in the stage of portfolio analysis varies. While variance can be minimized using a quadratic problem and CVaR can be minimized using a linear problem following Rockafellar and Uryasev (2000), the use of VaR leads to a non-convex NP-hard optimization problem.

In order to evaluate convergence of multiobjective genetic algorithm, we formulate and compute mean-variance, mean-VaR and mean-CVaR as multiobjective problems, and also mean-variance model as a quadratic problem and mean-CVaR model as a linear problem.

An important property of the variance of the portfolio return is that it allows defining the objective function as a quadratic function. The mean-variance model can be formulated as follows:

\[ \text{Model 1A} \]
\[
\begin{align*}
\text{Min } & \sigma^2(w) = w'\Omega w \\
\text{s.t.} & \quad w'r \geq r^* \\
& \quad w'1 = 1 \\
& \quad w_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*}
\]

\[ \text{Model 1B} \]
\[
\begin{align*}
\text{Min } & \sigma^2(w) = w'\Omega w \\
\text{s.t.} & \quad \text{Max } \text{E}(r) = w'r \\
& \quad w'1 = 1 \\
& \quad w_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*}
\]

where \( w \) is a vector containing the percentage of the budget invested in each asset, \( \Omega \) is the covariance matrix and \( r \) is a vector containing the expected rate of return of each asset. Model 1A is the classical quadratic problem whose exact solution is a function of the expected return fixed by the investor, \( r^* \), while Model 1B is the multiobjective problem that we solve with a multiobjective genetic algorithm.

The mean-VaR model uses the concept of VaR as a measure of risk instead of variance. Then, efficient portfolios are the solution of the following problem:

\[ \text{Model 2} \]
\[
\begin{align*}
\text{Min } & \text{VaR}_\alpha(w) = \inf \{ r | \text{F}(r) \geq \alpha \} \\
\text{s.t.} & \quad w'r = \text{E}(r) \\
& \quad w'1 = 1 \\
& \quad w_j \geq 0 \quad \forall j = 1, \ldots, n
\end{align*}
\]

VaR is difficult to optimize for discrete distributions since it is non-convex and has multiple local extrema. Mostly, approaches rely on linear approximation of the portfolio risks and assume a joint normal distribution of the underlying parameters (Jorion, 1996, Duffie and Pan, 1997). Optimization requires smoothing or heuristic techniques as those presented in Gaivoronski and Pflug (2005) or Gilli et al. (2006). We use a multiobjective genetic algo-
rhythm which does not rely on specific assumptions about the distribution of the portfolio return.

As we pointed out previously, CVaR can be defined as the conditional expected loss under the condition that it exceeds VaR. For general distributions, CVaR is more attractive than VaR since it is sub-additive and convex. Hence, the problem of minimizing CVaR for finding efficient portfolios is convex. CVaR of a random variable \( r(w, y) \), which represents the return of the portfolio \( w \) under scenarios \( y \), can be calculated by solving a convex optimization problem (Rockafellar and Uryasev, 2000, 2002). The approach characterises \( \text{VaR}_\alpha(w) \) and \( \text{CVaR}_\alpha(w) \) in terms of the function \( F_\alpha(w, \text{VaR}) \).

\[
F_\alpha(w, \text{VaR}) = -\text{VaR} + \frac{1}{\alpha} \int [-r(w, y) + \text{VaR}]^+ p(y) dy
\]  

(7)

Where \([u]^+ = u \) for \( u \geq 0 \) and \([u]^+ = 0 \) for \( u \leq 0 \). As a function of \( \text{VaR}_\alpha(w) \), \( F_\alpha(w, \text{VaR}) \) is convex and continuously differentiable and \( \text{CVaR}_\alpha(w) \) can be determined minimizing \( F_\alpha(w, \text{VaR}) \), that is, \( \text{CVaR}_\alpha(w) = \min F_\alpha(w, \text{VaR}) \). The integral in (7) can be approximated by sampling the probability distribution of \( y \) according to its density \( p(y) \). If sampling generates \( T \) scenarios, \( y_j, j = 1, \ldots, T \) then the approximation is,

\[
F_\alpha(w, \text{VaR}) = -\text{VaR} + \frac{1}{\alpha} \sum_{j=1}^{T} \pi_j [-r(w, y_j) + \text{VaR}]^+
\]  

(8)

where \( \pi_j \) is the probability of scenarios \( y_j \) and \( r(w, y_j) = \sum_{i=1}^{n} w_j r_{ij} \), being \( r_{ij} \) the return of asset \( i \) under scenario \( j \). Using auxiliaries variables \( z_j, j = 1, \ldots, T \) the function \( F_\alpha(w, \text{VaR}) \) can be replaced by the linear function, \( F_\alpha(w, \text{VaR}) = -\text{VaR} + \frac{1}{\alpha} \sum_{j=1}^{T} \pi_j z_j \) and the set of linear constraints, \( z_j \geq -r(w, y_j) + \text{VaR}, z_j \geq 0, j = 1, \ldots, T \). Then investors have to solve the linear problem represented in model 3A. Lim et al. (2008) propose an approach to reduce the time consumed for solving the linear problem under a large number of scenarios.

**Model 3A**

\[
\begin{align*}
\text{MinCVaR}_\alpha(w) &= \text{Min} \text{ VaR} + \frac{1}{\alpha T} \sum_{j=1}^{T} z_j \\
\text{s. t.} \\
z_j \geq -\sum_{j=1}^{T} w_j r_{ij} + \text{VaR} & \forall j = 1, \ldots, T \\
z_j \geq 0 & \forall j = 1, \ldots, T \\
w'1 &= 1 \\
w' r \geq r^* \\
w_j \geq 0 & \forall j = 1, \ldots, n
\end{align*}
\]

**Model 3B**

\[
\begin{align*}
\text{Max E}(r) &= w'r \\
\text{s. t.} \\
w'1 &= 1 \\
w_j \geq 0 & \forall j = 1, \ldots, n
\end{align*}
\]

(9)
Model 3A is a linear problem with an exact solution for each \( r^* \) and Model 3B is the formulation used in the multiobjective algorithm problem. In this case we do not need linearization since a multiobjective genetic algorithm searches the space of solutions.

Solving Model 1 (A or B), 2 and 3 (A or B) allows us to obtain sets of optimal portfolios. The question is how different the efficient portfolios obtained with different risk measures are. Are portfolios that are efficient in the mean variance model (Model 1A or 1B) efficient in mean-VaR or mean-CVaR space or not? If the answer is yes or next to yes, given that increasing the number of risk measures implies defining the investor’s preferences through more complex utility functions, then we should use only one risk measure. However, if the answer is not, that is, if each risk measure reflects different risk dimensions of a portfolio then we should consider the differences between the efficient frontiers increasing the number of risk measures. In this case it is necessary to formulate the selection problem as a multiobjective problem with two or more risk measures to minimize.

### 2.3. Portfolio selection under several risk measures

The inclusion of more than one risk measure to obtain efficient portfolios has been approached in the literature from two perspectives: introducing additional constrains or introducing additional objective functions as multiobjective problems. Early examples on using an additional variable for the mean-variance model can be found in Konno et al. (1993) in which mean, absolute deviation and skewness are combined. The skewness is maximized subject to constraints in mean and absolute deviation. Konno and Suzuki (1995) show an efficient algorithm to optimize a mean-variance-skewness model. Alexander et al. (2007) obtain efficient portfolios under mean-variance model when VaR or CVaR are used as constraints. A pure multiobjective proposal can be found in Roman et al. (2007) where a mean-variance-CVaR model is proposed and an optimization approach is given. Introducing several risk measures provides more information about portfolio risk. Although the asset return was non-normal distributed, portfolio return follows other unknown distribution and efficient portfolios are a subset of all feasible portfolios and their unknown distribution can behave similar to the normal distribution. Under normal distributions of returns the three measures provide the same optimal portfolio (Rockafellar and Uryasev, 2000) and a multiobjective risk is worthless. However, for skewed distributions, VaR and CVaR optimal portfolios may be quite different.

A multiobjective problem in general is expressed as \( \text{Max} \{ f_1(w), f_2(w), \ldots, f_n(w) \} \). In the classical asset allocation problem (model 1B) there are two objectives to optimize: mean and variance. If an additional risk measure is introduced the concept of efficient portfolio is the same but the problem has a third dimension. A feasible solution \( w^* \) Pareto dominates another feasible solution \( w' \) if \( f_i(w^*) \geq f_i(w') \) for all \( i \) with at least one strict inequality. For instance, in a mean-VaR-CVaR model the efficient portfolios are the Pareto efficient solutions of a multiobjective problem in which the expected value is maximized while the VaR and CVaR are minimized. The problem may be formulated as,
Model 4

\[ \text{Max } E(r) = w'r \]
\[ \text{Min VaR}_\alpha (w) = \inf \{ r | F(r) \geq \alpha \} \]
\[ \text{Min CVaR}_\alpha (w) = \frac{1}{\alpha T} \sum_{j=1}^{\alpha} R_{p,j} \]

s. t.
\[ w'1 = 1 \]
\[ w_j \geq 0 \forall j = 1, \ldots, n \]

Minimizing VaR does not control losses exceeding VaR since it does not take into account the shape of the tail. Then, introducing CVaR as a second measure of risk captures these losses. In mean-VaR-CVaR model (Model 4), \( w^* \) is preferred to portfolio \( w' \) if and only if \( E(r_{w^*}) \geq E(r_{w'}) \), \( \text{VaR}_\alpha (w^*) \leq \text{VaR}_\alpha (w') \), and \( \text{CVaR}_\alpha (w^*) \leq \text{CVaR}_\alpha (w') \) with at least one strict inequality. When we consider three objectives we obtain a surface of efficient portfolios instead of a line. One issue addressed in this paper is how the efficient frontier changes when we increase the measures of risk.

On the other hand, in a mean-variance-VaR model the efficient portfolios are the Pareto efficient solutions of:

Model 5

\[ \text{Max } E(r) = w'r \]
\[ \text{Min } \sigma^2(w) = w' \Omega w \]
\[ \text{Min } \text{VaR}_\alpha (w) = \inf \{ r | F(r) \geq \alpha \} \]

s. t.
\[ w'1 = 1 \]
\[ w_j \geq 0 \forall j = 1, \ldots, n \]

This paper is related to Roman et al. (2007), which study efficient portfolios considering more than one measure of risk as objective function. The main difference of our contribution is that we present a practical optimization approach where VaR is considered as a measure of risk in a multiobjective problem. In addition, we evaluate if the inclusion of several risk measures is needed or irrelevant for efficient portfolios. As Roman et al. (2007) indicates VaR leads to complex problems, whose solution requires the development of new algorithms. Roman et al. (2007) only use mean-variance-CVaR model while we focus on mean-VaR-CVaR (Model 4) and mean-variance-VaR (Model 5) models.
3. OPTIMIZATION APPROACH: MULTIOBJECTIVE GENETIC ALGORITHM

By definition, the portfolio optimization problem proposed by Markowitz (1952) is multi-objective with two conflicting criteria: maximising the return and minimising the level of risk. Introducing several risk measures increases the number of objectives. Multi-objective optimization problems do not have a single solution but a set of solutions equally optimal that define the efficient frontier (or Pareto-optimal front).

Genetic algorithms (GAs) (see, Holland, 1975, Goldberg, 1989) are stochastic optimization techniques that mimic the way species evolve in nature. GAs emulate this process by encoding the points of the search space (called individuals) in a chromosome-like shape and evolving a population of them through a number of generations using mechanisms drawn from natural evolution. Along the evolution the individuals’ fitness is evaluated according to how well they solve the problem at hand. The better the fitness of an individual, the more chances it has to produce offspring for the next generation. As the generations progress, it results in the prevalence of stronger solution over weaker ones. Thus, the evolution process tends to near optimal solutions.

A GA initiates the process of searching by randomly generating an initial population of possible solutions. The performance of each solution is evaluated using a fitness function, which is a measure of how good the performance of the solution is. Then, a new generation is produced according to the three main operators of the GA: selection, crossover and mutation.

Selection determines which solutions are chosen for mating according to the principal of survival of the fittest (i.e. the better the performance of the solution, the more likely it is to be chosen for mating and therefore the more offspring it produces). In this work we used tournament selection. The tournament selection method works by choosing a group of q individuals randomly from the population and selecting the best individual in terms of fitness from this group.

Crossover allows an improvement in the species in terms of the evolution of new solutions that are fitter than any seen before. The crossover operator combines the features of two parents to create new solutions. One or several crossover points are selected at random on each parent and then, complementary fractions from the two parents are spliced together to form a new chromosome, as shown in Figure 1.

Mutation reintroduces values that might have been lost through selection or crossover, or creates totally new features. The mutation operator alters a copy of a chromosome. One or more locations are selected on the chromosome and replaced with new randomly generated values. Mutation is used to help ensure that all areas of the search space remain reachable providing higher variation in the chromosomes of each population. It also allows the reintroduction of features that might have been lost during the selection procedure.

The cycle selection-crossover-mutation-evaluation is performed until a termination criterion is met (for instance, a predetermined number of generations) (see Figure 2).
For multi-objective GA (Coello, 2006) the concept of fitness changes. The fitness of an individual it is not anymore how well it solves the problem, but it is based on the Pareto optimality concept. The fitness of an individual is a function of how many individuals it dominates and by how many individuals it is dominated. Thus, nondominated individuals have the highest possible fitness and the rest of individuals are ranked according to their dominance relations.

Other important concept that is often used in GAs is the concept of elitism. In an elitist selection technique the best individuals of the population are automatically selected to go to the next generation without undergoing crossover or mutation. In the context of multiobjective GAs, the use of a subpopulation (usually called archive) where the nondominated individuals are stored along the generations guarantees that nondominated solutions are
not lost during the run and that a solution reported as nondominated is nondominated with respect to any other solution generated by our algorithm.

3.1. GA IMPLEMENTATION

The GA implementation used in this work is based on ECJ\(^{(1)}\), a research evolutionary computation system in Java developed at George Mason University’s Evolutionary Computation Laboratory (ECLab). For the multi-objective aspect of the optimization the SPEA2 (Strength Pareto Evolutionary Algorithm 2) package of ECJ was used (Zitzler et al. 2001). SPEA2 is an improved version of SPEA which incorporates a fine-grained fitness assignment strategy, a density estimation technique and an enhanced archive truncation method. As most of the multi-objective evolutionary methods it keeps an archive where the non-dominated solutions are stored. The size of the archive is set by the user so that if the number of non-dominated solutions is bigger than the archive size the archive is truncated.

The algorithm works as follows:

- In Step 1 and 2 the archive, \(A(g)\), where the non-dominated solutions are stored and the population, \(P(g)\), are initialized. \(A(0)\) is an empty set and \(P(0)\) is initialized at random.
- In Step 3 the generation counter \(g\) is set to 1 and then the evolution loop starts.
- In Step 4 and 5 the individuals in the population and the archive are evaluated.
- According to this evaluation a new archive is created in Step 6 containing all the non-dominated individuals found in the union of the previous archive and the population.
- If the size of the resulting archive exceeds the archive size, in Step 7 the archive is truncated. This truncation method removes those individuals which are at the minimum distance of another individual. This way the characteristics of the non-dominated front are preserved and outer solutions are not lost.
- The termination criterion in step 8 stops the algorithm when the number of generations has been completed.
- In Step 9 tournament selection with replacement is performed in the archive set in order to fill the mating pool, \(M(g)\).
- The new population, \(P(g)\), is created in Step 10 by applying crossover and mutation to the mating pool.
- In Step 11 the generation counter is increased.

The control parameters of the GA used are quite standard. The GA is generational. It uses tournament selection with tournament size of 7. The probabilities of crossover and mutation are 1 and 0.05 respectively. The population size is 5000 and the archive size is 100. The run finishes after 100 generations. Each individual is encoded as a vector of integers ranging from 0 to 99. Every element of the vector represents the percentage of the budget invested in that particular asset \((w_{j}^{GA} \leq 0 \ j = 1, \ldots, n)\). Therefore, the length of the vector equals the number of assets available in the portfolio. However, the summation of these

\(^{(1)}\) http://cs.gmu.edu/~eclab/projects/ecj/
weights will not be 1, violating the constraint \( \sum_{j=1}^{n} w_j = 1 \). This constraint imposes the need of normalizing the vector during the decoding process as follows:

\[
    w_j^{GA} = \frac{w_j^{GA}}{\sum_{j=1}^{n} w_j^{GA}}
\]  

(12)

where \( w_j \) represents the weight invested in asset \( j \). However, these normalized weights are real values.

4. DATA AND EMPIRICAL ANALYSIS

The data used in this work were extracted from the Bloomberg database. It is a set composed of fifty stocks which belonged to the Eurostoxx 50 index in January 2008. Three stocks with negative expected return in the analysis period were eliminated. We use daily data of these stocks from January 2003 to December 2007. This gives us 1300 observations per stock. We chose daily data instead of monthly data to avoid inaccurate \( \text{VaR} \) estimates from small samples.

<table>
<thead>
<tr>
<th>Country</th>
<th>Company</th>
<th>Mean</th>
<th>SD</th>
<th>VaR(_{95%})</th>
<th>CVaR(_{95%})</th>
<th>BJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>France</td>
<td>Air Liquide</td>
<td>0.048</td>
<td>1.270</td>
<td>1.928</td>
<td>2.800</td>
<td>175.6</td>
</tr>
<tr>
<td>France</td>
<td>Alcatel-Lucent</td>
<td>0.004</td>
<td>2.333</td>
<td>3.472</td>
<td>5.457</td>
<td>640.5</td>
</tr>
<tr>
<td>Germany</td>
<td>Allianz, S.E.</td>
<td>0.039</td>
<td>1.853</td>
<td>2.911</td>
<td>4.398</td>
<td>985.6</td>
</tr>
<tr>
<td>Italy</td>
<td>Assicurazioni Generali SpA</td>
<td>0.041</td>
<td>1.243</td>
<td>1.771</td>
<td>2.832</td>
<td>308.1</td>
</tr>
<tr>
<td>France</td>
<td>AXA, S.A.</td>
<td>0.054</td>
<td>1.861</td>
<td>2.898</td>
<td>4.238</td>
<td>344.2</td>
</tr>
<tr>
<td>Spain</td>
<td>BBVA, S.A.</td>
<td>0.043</td>
<td>1.349</td>
<td>2.141</td>
<td>3.126</td>
<td>213.9</td>
</tr>
<tr>
<td>Spain</td>
<td>Banco Santander, S.A.</td>
<td>0.059</td>
<td>1.360</td>
<td>2.188</td>
<td>3.202</td>
<td>300.6</td>
</tr>
<tr>
<td>Germany</td>
<td>BASF, S.E.</td>
<td>0.074</td>
<td>1.429</td>
<td>2.061</td>
<td>3.087</td>
<td>623.4</td>
</tr>
<tr>
<td>Germany</td>
<td>Bayer, AG</td>
<td>0.087</td>
<td>1.993</td>
<td>2.551</td>
<td>4.161</td>
<td>3557.7</td>
</tr>
<tr>
<td>France</td>
<td>BNP Paribas</td>
<td>0.047</td>
<td>1.487</td>
<td>2.371</td>
<td>3.260</td>
<td>425.4</td>
</tr>
<tr>
<td>France</td>
<td>Carrefour, S.A.</td>
<td>0.015</td>
<td>1.432</td>
<td>2.171</td>
<td>3.254</td>
<td>1700.9</td>
</tr>
<tr>
<td>France</td>
<td>Cie de Saint-Gobain</td>
<td>0.060</td>
<td>1.608</td>
<td>2.580</td>
<td>3.629</td>
<td>2694.3</td>
</tr>
<tr>
<td>France</td>
<td>Credit Agricole, S.A.</td>
<td>0.036</td>
<td>1.485</td>
<td>2.210</td>
<td>3.322</td>
<td>112.9</td>
</tr>
<tr>
<td>Germany</td>
<td>Daimler, AG</td>
<td>0.056</td>
<td>1.650</td>
<td>2.717</td>
<td>3.630</td>
<td>24340.5</td>
</tr>
<tr>
<td>Germany</td>
<td>Deutsche Bank, AG</td>
<td>0.048</td>
<td>1.559</td>
<td>2.494</td>
<td>3.415</td>
<td>1227.9</td>
</tr>
<tr>
<td>Germany</td>
<td>Deutsche Boerse, AG</td>
<td>0.151</td>
<td>1.669</td>
<td>2.434</td>
<td>3.577</td>
<td>2190.6</td>
</tr>
</tbody>
</table>

(Continúa pág. sig.)
### Table 1 (cont.)

#### SUMMARY OF DATA STATISTICS

<table>
<thead>
<tr>
<th>Country</th>
<th>Company</th>
<th>Mean</th>
<th>SD</th>
<th>VaR&lt;sub&gt;95%&lt;/sub&gt;</th>
<th>CVaR&lt;sub&gt;95%&lt;/sub&gt;</th>
<th>BJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Germany</td>
<td>Deutsche Telekom, AG</td>
<td>0.011</td>
<td>1.465</td>
<td>2.063</td>
<td>3.529</td>
<td>20488.1</td>
</tr>
<tr>
<td>Germany</td>
<td>E.ON, AG</td>
<td>0.099</td>
<td>1.418</td>
<td>2.074</td>
<td>3.163</td>
<td>6325.7</td>
</tr>
<tr>
<td>Italy</td>
<td>Enel, SpA</td>
<td>0.042</td>
<td>0.973</td>
<td>1.449</td>
<td>2.237</td>
<td>355.6</td>
</tr>
<tr>
<td>Italy</td>
<td>ENI, SpA</td>
<td>0.036</td>
<td>1.147</td>
<td>1.855</td>
<td>2.679</td>
<td>918.8</td>
</tr>
<tr>
<td>Belgium</td>
<td>Fortis</td>
<td>0.016</td>
<td>1.770</td>
<td>2.634</td>
<td>4.339</td>
<td>678.4</td>
</tr>
<tr>
<td>France</td>
<td>France Telecom, S.A.</td>
<td>0.036</td>
<td>1.641</td>
<td>2.339</td>
<td>3.621</td>
<td>238.27</td>
</tr>
<tr>
<td>France</td>
<td>Groupe Danone</td>
<td>0.049</td>
<td>1.222</td>
<td>1.705</td>
<td>2.470</td>
<td>1191.2</td>
</tr>
<tr>
<td>Spain</td>
<td>Iberdrola, S.A.</td>
<td>0.087</td>
<td>1.094</td>
<td>1.523</td>
<td>2.277</td>
<td>3887.1</td>
</tr>
<tr>
<td>Holland</td>
<td>ING Groep NV</td>
<td>0.035</td>
<td>1.851</td>
<td>2.827</td>
<td>4.389</td>
<td>8069.3</td>
</tr>
<tr>
<td>Italy</td>
<td>Intesa Sanpaolo, SpA</td>
<td>0.070</td>
<td>1.535</td>
<td>2.191</td>
<td>3.323</td>
<td>147.1</td>
</tr>
<tr>
<td>Holland</td>
<td>Koninklijke Philips Electronics, NV</td>
<td>0.037</td>
<td>1.828</td>
<td>2.800</td>
<td>4.075</td>
<td>548.9</td>
</tr>
<tr>
<td>France</td>
<td>L’Oreal, S.A.</td>
<td>0.021</td>
<td>1.347</td>
<td>2.021</td>
<td>3.032</td>
<td>871.2</td>
</tr>
<tr>
<td>France</td>
<td>LVMH, S.A.</td>
<td>0.054</td>
<td>1.369</td>
<td>2.166</td>
<td>2.984</td>
<td>1032.8</td>
</tr>
<tr>
<td>Germany</td>
<td>Muenchener Rueckversicherungs AG</td>
<td>0.011</td>
<td>1.845</td>
<td>2.738</td>
<td>4.533</td>
<td>2398.5</td>
</tr>
<tr>
<td>Finland</td>
<td>Nokia, OYJ</td>
<td>0.038</td>
<td>2.019</td>
<td>2.887</td>
<td>4.755</td>
<td>170.7</td>
</tr>
<tr>
<td>France</td>
<td>Renault, S.A.</td>
<td>0.056</td>
<td>1.721</td>
<td>2.745</td>
<td>3.842</td>
<td>459.7</td>
</tr>
<tr>
<td>Spain</td>
<td>Repsol YPF, S.A.</td>
<td>0.048</td>
<td>1.230</td>
<td>1.923</td>
<td>2.857</td>
<td>435.1</td>
</tr>
<tr>
<td>Germany</td>
<td>RWE, AG</td>
<td>0.100</td>
<td>1.535</td>
<td>2.273</td>
<td>3.456</td>
<td>944.7</td>
</tr>
<tr>
<td>France</td>
<td>Sanofi-Aventis, S.A.</td>
<td>0.005</td>
<td>1.471</td>
<td>2.235</td>
<td>3.426</td>
<td>7508.2</td>
</tr>
<tr>
<td>Germany</td>
<td>SAP, AG</td>
<td>0.043</td>
<td>1.719</td>
<td>2.430</td>
<td>3.798</td>
<td>539.9</td>
</tr>
<tr>
<td>Germany</td>
<td>Schneider Electric, SA</td>
<td>0.054</td>
<td>1.442</td>
<td>2.258</td>
<td>3.241</td>
<td>59.29</td>
</tr>
<tr>
<td>Germany</td>
<td>Siemens, AG</td>
<td>0.069</td>
<td>1.617</td>
<td>2.535</td>
<td>3.569</td>
<td>629.3</td>
</tr>
<tr>
<td>France</td>
<td>Societe Generale</td>
<td>0.042</td>
<td>1.550</td>
<td>2.416</td>
<td>3.537</td>
<td>6005.0</td>
</tr>
<tr>
<td>France</td>
<td>Suez, SA</td>
<td>0.076</td>
<td>1.952</td>
<td>2.657</td>
<td>4.582</td>
<td>668.3</td>
</tr>
<tr>
<td>Spain</td>
<td>Telefonica, SA</td>
<td>0.077</td>
<td>1.198</td>
<td>1.853</td>
<td>2.664</td>
<td>2330.4</td>
</tr>
<tr>
<td>France</td>
<td>Total, SA</td>
<td>0.038</td>
<td>1.240</td>
<td>2.143</td>
<td>2.774</td>
<td>339.6</td>
</tr>
<tr>
<td>Italy</td>
<td>UniCredit SpA</td>
<td>0.029</td>
<td>1.273</td>
<td>2.007</td>
<td>3.033</td>
<td>3106.8</td>
</tr>
<tr>
<td>Holland</td>
<td>Unilever NV</td>
<td>0.019</td>
<td>1.227</td>
<td>1.776</td>
<td>2.942</td>
<td>399.8</td>
</tr>
<tr>
<td>France</td>
<td>Vinci, SA</td>
<td>0.103</td>
<td>1.404</td>
<td>1.947</td>
<td>2.919</td>
<td>555.3</td>
</tr>
<tr>
<td>France</td>
<td>Vivendi</td>
<td>0.050</td>
<td>1.639</td>
<td>2.439</td>
<td>3.792</td>
<td>2778.3</td>
</tr>
<tr>
<td>Germany</td>
<td>Volkswagen, AG</td>
<td>0.107</td>
<td>1.773</td>
<td>2.717</td>
<td>3.789</td>
<td>200813.2</td>
</tr>
</tbody>
</table>

*Mean:* the average daily return. *SD:* standard deviation. *VaR<sub>95%</sub>:* the 95% 1-day VaR, and *CVaR<sub>95%</sub>:* the 95% 1-day CVaR are expressed in percentage and are computed with daily returns from January 2003 to December 2007. *BJ* is the Bera-Jarque statistic.
Table 1 reports the descriptive analysis of the data identifying companies by country. It can be observed that the mean daily return is close to zero. This is consistent with computing VaR under the assumption of expected daily return equal to zero. Table 1 reports standard deviation, \( \text{VaR} \) and \( \text{CVaR} \) at 95% confidence level for each stock. In order to compute the three risk measures we used historical simulation as mentioned previously (see Section 2.1).

Additionally, we compute Bera-Jarque (BJ) statistic to test normality. The BJ statistic has a chi-squared distribution with two degrees of freedom under the null hypothesis that returns are normally distributed. As can be observed the minimum value of the BJ statistic is 59.29 while the critical value is 9.21 for a 1% significance level. Hence normality is rejected in all cases and portfolio risk level ranking is different depending on the measure selected: standard deviation, \( \text{VaR} \) or \( \text{CVaR} \). Also, Table 1 reports standard deviation, \( \text{VaR} \) and \( \text{CVaR} \) at 95% confidence level. For example, we can find Siemens AG with 1.617% of deviation, 2.535% of \( \text{VaR} \) and 3.569% of \( \text{CVaR} \) while SAP AG has higher deviation 1.719%, lower \( \text{VaR} \) 2.43% and higher \( \text{CVaR} \) 3.798%. These are only examples since they are not representative of the subset of efficient portfolios.

Although it is no trivial compare risk and return of efficient portfolios since the concept of efficiency is respect to a determine risk measure we show, through figures and tables, the differences between portfolios obtaining subsets of efficient portfolios under different risk measures (Model 1, 2 and 3) and translating them to different spaces. Furthermore, we previously evaluate the quality of the solution, subset of efficient portfolios, obtained with multiobjective genetic algorithm by comparing it with the solution obtained with an exact algorithm (Model 1A versus 1B, Model 3A versus 3B).

For testing the reliability of the multi-objective evolutionary approach (GA), we solved the classical mean-variance problem using GAs, Model 1B, and we compared the results obtained with those of the classical Quadratic Program (QP), Model 1A. Also, we solved the mean-CVaR problem using GAs, Model 3B, and we compared the results obtained with those of linear program (LP) proposed by Rockefellar and Uryasev (2000, 2002), Model 3A.

### Table 2

**Differences in deviation and CVaR among optimal portfolios calculated using different optimization techniques**

<table>
<thead>
<tr>
<th></th>
<th>( w^{GA} - w^{QP} )</th>
<th>( w_{\text{CVaR}}^{GA} - w_{\text{CVaR}}^{LP} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.0049</td>
<td>0.0190</td>
</tr>
<tr>
<td>Deviation</td>
<td>0.0027</td>
<td>0.0081</td>
</tr>
<tr>
<td>Min.</td>
<td>0.0014</td>
<td>0.0092</td>
</tr>
<tr>
<td>Max.</td>
<td>0.0196</td>
<td>0.0430</td>
</tr>
</tbody>
</table>

**Note:** values are expressed in percentage.

Figure 3 illustrates that the \( \sigma \)-efficient frontiers calculated using GAs and QP (Model 1A and 1B) and CVaR-efficient frontiers calculated using GAs and LP (Model 3A and 3B) are overlapped. Also, it is showed the \( \text{VaR} \)-efficient frontier obtained using GAs (Model 2). In this case, there is not an exact algorithm available and heuristic methods are essential.
As it is showed the efficient frontier irregularity reflects that VaR is not a convex risk measure.

** FIGURE 3 **
** MEAN-VARIANCE, MEAN-CVaR AND MEAN-VaR EFFICIENT FRONTIERS COMPUTED WITH DIFFERENT TECHNIQUES **

Mean-variance efficient frontier was computed using multiobjective GA and quadratic programming, QP. Mean-CVaR efficient frontier was computed using multiobjective GA and linear programming, LP. Mean-VaR efficient frontier was computed using multiobjective GA.
Table 2 reports the differences between efficient frontiers. For each level of return, \( w^{GA} - w^{QP} \) is the difference in standard deviation between a portfolio obtained with GA (Model 1B) and QP (Model 1A). The mean error of the \( \sigma \)-efficient portfolios computed using QP (Model 1A) and GA (Model 1B) is close to zero (0.0049%). The efficient portfolio with the highest difference is 0.0196% which further confirms the good results provided by the multiobjective genetic algorithm to minimize deviation and maximize return. Similar conclusions are obtained when we maximize return and minimize CVaR (Model 3B) and we compare these results of the linear programming applying the transformation of the objective function proposed by Rockefellar and Uryasev (2000, 2002), (Model 3A). For each level of return, \( w^{CVaR}_{\text{GA}} - w^{CVaR}_{\text{LP}} \) is the difference in CVaR value between a portfolio obtained with multiobjective GA (Model 3B) and LP (Model 3A). The mean error of the CVaR-efficient portfolios computed using LP and GA is close to zero (0.0190%), the efficient portfolio with the highest difference is 0.0430%.

So far we have proven that multiobjective GAs can easily solve Markowitz classical problem and the mean-CVaR problem without linear transformation. We want to quantify how different these risk measures (deviation, CVaR and VaR) are in the subset of efficient portfolios to know if this risk measures are interchangeable. Of course, as investors choose a portfolio in the subset of efficient portfolios to know if a risk measure is interchangeable in the set of feasible portfolios is irrelevant.

To deal with the task of comparing efficient frontier under different risk measures, we use the efficient portfolios obtained under the same algorithm (Model 1B, 2 and 3B). For each efficient portfolio under a determined risk measure we compute the portfolio risk applying the other two definitions of risk not used to determine that it was an efficient portfolio. For example, for each efficient portfolio obtained from mean-variance model (Model 1B), named \( \sigma \)-efficient portfolio, we compute its VaR and CVaR in order to compare them with the efficient portfolios obtained from mean-VaR Model (Model 2), named VaR-efficient portfolios, and with the efficient portfolios obtained from mean-CVaR model (Model 3B), named CVaR-efficient portfolios.

When investors use Model 1, 2 or 3, they choose an efficient portfolio under one risk measure and ignore the other measures of risk. It is not wrong when we are assuming that the used risk measure and the ignored are substitutive, that is, all the risk measures give similar efficient frontier. Then the question is: deviation of \( \sigma \)-efficient portfolios (Model 1B), deviation of VaR-efficient portfolios (Model 2) and deviation of CVaR-efficient portfolios (Model 3B) are equivalent? Alternatively, VaR of \( \sigma \)-efficient portfolios (Model 1B), VaR of VaR-efficient portfolios (Model 2) and VaR of CVaR-efficient portfolios (Model 3B) are equivalent? And finally, CVaR of \( \sigma \)-efficient portfolios (Model 1B), CVaR of VaR-efficient portfolios (Model 2) and CVaR of CVaR-efficient portfolios (Model 3B) are equivalent? We only know that deviation of \( \sigma \)-efficient portfolios (Model 1B), VaR of VaR-efficient portfolios (Model 2) and CVaR of CVaR-efficient portfolios (Model 3B) are the lowest. Graphically, it can be answered representing the portfolios in different spaces, (mean-variance space, mean-VaR space and mean-CVaR space), Figure 4. If the three risk measures are substituted, then the efficient frontiers should be overlapped in the three spaces and investors could consider whatever one in the portfolio selection model choosing between mean-variance, mean-VaR
or mean-CVaR model (Model 1, 2 or 3). Of course the investors’ election could be the simplest or better-known portfolio selection model (mean-variance model).

On the other hand, if the three risk measures are complementary, then the efficient frontiers are not overlapped in the three spaces. In this case, it is worthwhile to include more than one risk measure in the objective function. Investors should define their preferences in terms of the different risk measures and select a portfolio from the mean-variance-VaR-CVaR model since all the risk measures are jointly considered for obtaining the efficient portfolios.

**Figure 4**

**Mean-variance, Mean-CVaR and Mean-VaR efficient frontiers represented in Mean-variance, Mean-CVaR and Mean-VaR spaces**

*SD line represents σ-optimal portfolios, VaR line repents VaR-optimal portfolios and CVaR line represents CVaR-optimal portfolios.*
Table 3

Differences in deviation and CVaR among optimal portfolios under different risk measures

<table>
<thead>
<tr>
<th></th>
<th>SD differences</th>
<th>CVaR differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w_{\sigma} - w_{\text{CVaR}}$</td>
<td>$w_{\sigma} - w_{\text{VaR}}$</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0080</td>
<td>0.0314</td>
</tr>
<tr>
<td>Deviation</td>
<td>0.0055</td>
<td>0.0095</td>
</tr>
<tr>
<td>Min</td>
<td>0.0020</td>
<td>0.0085</td>
</tr>
<tr>
<td>Max</td>
<td>0.0237</td>
<td>0.0630</td>
</tr>
</tbody>
</table>

Note: Differences in deviation and CVaR are expressed in percentage. $w_{\sigma}$, $w_{\text{CVaR}}$ and $w_{\text{VaR}}$ represent optimal portfolios under deviation, CVaR and VaR risk measures, respectively.

Figure 4 illustrates differences between $\sigma$-efficient portfolios, CVaR-efficient portfolios and VaR-efficient portfolios in mean-deviation, mean-CVaR and mean-VaR spaces. As can be observed, even though in each space the efficient portfolios are those obtained as optimal for the risk measure that define the space, $\sigma$-efficient portfolios and CVaR-efficient portfolios are close in all spaces while VaR-efficient portfolios are more separated in all spaces. This confirms that obtaining efficient portfolios for several risk measures is not always so important. Concretely, when the risk measures considered are standard deviation and CVaR the efficient portfolios are similar. However, when VaR is considered as risk measure, the inclusion of other risk measure such as CVaR or variance gives extra information to the investor.

Table 3 quantifies the difference in deviation and CVaR between efficient portfolios under different risk measures. The mean difference in terms of portfolio standard deviation between $\sigma$-efficient portfolios, named $w_{\sigma}$ (Model 1B) and CVaR-efficient portfolios, named $w_{\text{CVaR}}$ (Model 3B) is 0.008% with a maximum difference of 0.0237% while the mean difference with VaR-efficient portfolios, named $w_{\text{VaR}}$ (Model 2) is 0.0314% with a maximum difference of 0.063%. This means that deviation is a measure more related to CVaR than VaR. With respect to CVaR, the mean difference between the CVaR of CVaR-efficient portfolios (Model 3B) and the CVaR of $\sigma$-efficient portfolios (Model 1B) is 0.0201% with a maximum difference of 0.0531% while the mean difference with VaR-efficient portfolios (Model 2) is 0.0895% with a maximum difference of 0.2362%.

Figure 5 presents the efficient portfolios: $\sigma$-efficient portfolios (Model 1B), CVaR-efficient portfolios (Model 3B) and VaR-efficient portfolios (Model 2) are represented in the $\sigma$-CVaR, $\sigma$-VaR and VaR-CVaR spaces for different fixed returns. The differences between $\sigma$-efficient portfolios and CVaR-efficient portfolios in the $\sigma$-CVaR space are the smallest and the differences between VaR-efficient portfolios and CVaR-efficient portfolios in the VaR-CVaR space are the biggest as Table 3 reflects.

These results reflect a more similarity between $\sigma$-efficient portfolios (Model 1B) and CVaR-efficient portfolios (Model 3B) than between them and VaR-efficient portfolios (Model 2). These results indicate that it could be interesting for investors to incorporate VaR and other risk measure, variance or CVaR, in the portfolio optimization problem. Investors could choose between solving the mean-VaR-CVaR model (Model 4) or the mean-variance-VaR model...
These models 4 and 5 are multiobjective as models 1B, 2 and 3B, and they cannot be reduced to one objective function as it is done in model 1A and 3A by solving them for different values of $r^*$. But they can be solved easily applying a multiobjective genetic algorithm.

\textbf{FIGURE 5}

\textit{$\sigma$-optimal portfolios, CVaR-optimal portfolios and VaR-optimal portfolios represented in the $\sigma$-CVaR, $\sigma$-VaR and VaR-CVaR spaces}
It has to be pointed out that as $\sigma$-efficient portfolios are the efficient portfolios in mean-variance space they are also efficient in mean-variance-$VaR$ or mean-variance-$CVaR$ space. This is due to the fact that for a fixed return their volatility is the minimum. That is, other portfolios could have a smaller $CVaR$ or $VaR$ for the same return but not a smaller deviation. The same happens with $CVaR$-efficient portfolios. They are also efficient or non-dominated in mean-variance-$CVaR$ space. Therefore, mean-variance-$CVaR$ efficient portfolios are all the portfolios included in the area between $\sigma$-efficient portfolios and $CVaR$-efficient portfolios. As is shown in Figure 5 this surface is narrow, which means that the set of portfolios in which investors could select their portfolio selection is similar solving mean-variance model (Model 1A or 1B), mean-$CVaR$ model (Model 3A or 3B) or mean-variance-$CVaR$ model presented in Roman et al. (2007). However, mean-$VaR$-$CVaR$ and mean-variance-$VaR$ model (Model 4 and 5) offer wider surface of efficient portfolios to investors. Therefore, it is interesting to solve them since the solutions obtained offer a wide range of portfolios for the investors to choose according to their preferences. Similarly, taking into consideration the asset allocation with the three risk measures, that is, a mean-variance-$VaR$-$CVaR$ model, would be similar to consider mean-$VaR$-$CVaR$ or mean-variance-$VaR$ models since variance and $CVaR$ in the set of efficient portfolios generate a narrow surface.

### Table 4

<table>
<thead>
<tr>
<th>$r = 0.08%$</th>
<th>$r = 0.10%$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SD$</td>
<td>$CVaR$</td>
</tr>
<tr>
<td>Min $\sigma$</td>
<td>0.791*</td>
</tr>
<tr>
<td>Min $CVaR$</td>
<td>0.810</td>
</tr>
<tr>
<td>Min $VaR$</td>
<td>0.841</td>
</tr>
<tr>
<td>Min $VaR, \sigma$</td>
<td>0.805</td>
</tr>
<tr>
<td>Min $VaR, CVaR$</td>
<td>0.808</td>
</tr>
</tbody>
</table>

* Indicates minimum value for given return.

Table 4 illustrates the risk values of the $\sigma$-efficient portfolios (Model 1B), the $CVaR$-efficient portfolios (Model 3B), the $VaR$-efficient portfolios (Model 2), the $\sigma$-$VaR$-efficient portfolios (Model 5) and the $VaR$-$CVaR$-efficient portfolios (Model 4) for two level of expected return, 0.08% and 0.1%. Minimum values of $\sigma$, $VaR$ and $CVaR$ values are highlighted for each return. As can be observed, for each return $\sigma$-$VaR$-efficient portfolios have a higher deviation and lower $VaR$ than $\sigma$-efficient portfolios and lower deviation and higher $VaR$ than $VaR$-efficient portfolios. $\sigma$-$VaR$-efficient portfolios balance deviation and $VaR$ and graphically all the portfolios that make up the efficient surface are located in the gap shown in Figure 5 when $\sigma$-efficient portfolios (Model 1B) and $VaR$-efficient portfolios (Model 2) are represented in $\sigma$-$VaR$ space. The same happens when we analyze the $VaR$-$CVaR$-efficient portfolios for each level of return proposed. They have a lower $VaR$ and higher $CVaR$ than $CVaR$-efficient portfolios and higher $VaR$ and lower $CVaR$ than $VaR$-efficient portfolios. The surface of efficient portfolios for mean-$VaR$-$CVaR$ model is composed by portfolios in the gap between $VaR$-efficient portfolios and $CVaR$-efficient portfolios represented in the $VaR$-$CVaR$ space in Figure 5.
5. CONCLUSIONS

Variance, VaR and CVaR are different risk measures that capture different aspects of the return distribution related to risk. Usually asset allocation has been done through a multiobjective problem where expected return is maximized and one risk measure is minimized. To reduce one objective function, the problem is solved for proposed values of expected return. The inclusion of several risk measures is an issue that is being addressed recently by researchers. In this paper, we evaluate which combination of risk measures is more informative. In doing this, if the portfolio return is a normal distribution, variance, VaR and CVaR, generate the same efficient frontier since they are proportional. We used a historical simulation method to compute these risk measures in order to allow nonlinearities and avoid constraining conclusions to normal distribution assumption.

Investors decide about the risk measures considered to choose the investment portfolio. Based on our results we can conclude that investors do not need balance variance and CVaR when selecting an investment portfolio since efficient portfolios in the variance-CVaR space form a narrow surface. Then, ignoring a risk dimension does not change portfolio risk values significantly. If investor decide not ignoring them, after fixing a required return the range of variance and CVaR values to choose is too small. However, VaR efficient portfolios have a different level of variance and CVaR than variance efficient portfolios and CVaR efficient portfolios. As a result, we show that in the space variance-VaR and VaR-CVaR these portfolios generate a wider surface to balance investors’ preferences based on two risk measures.

Consequently, it is more informative to solve a mean-variance-VaR model or a mean-VaR-CVaR model since variance and VaR or VaR and CVaR behave differently and, as a consequence the resulting surface of efficient portfolios is significantly wider. Investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is VaR, for an expected return required and variance level ranging between the variance of the mean-variance efficient portfolios and the mean-VaR efficient portfolios. Alternatively and with similar results, investors can choose their optimal portfolio minimizing the maximum loss with a confidence level, that is VaR, for an expected return required and mean loss under the VaR level ranging between the CVaR of the mean-CVaR efficient portfolio and the mean-VaR efficient portfolio. In a general framework, where normality is not assumed, it is recommendable combining VaR with variance or CVaR in order to select an investment portfolio.

Finally, the results obtained show the feasibility of using multiobjective genetic algorithms for portfolio optimization problems under general risk definitions and several risk measures. The multiobjective genetic algorithm deals with the non-convexity characteristics of VaR and it solves the three-objectives models efficiently.

REFERENCES

ARTÍCULOS DOCTRINALES


